Fall 2024
14.12 Game Theory
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14.12 Final Exam

You have **3 hours** to complete the exam.

This exam is **closed book**. You may not use any electronic devices or written material brought into the exam.

The exam has **seven** questions worth a total of **120 points**:

- Problem 1: 10 points
- Problem 2: 15 points
- Problem 3: 20 points
- Problem 4: 15 points
- Problem 5: 20 points
- Problem 6: 15 points
- Problem 7: 25 points

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Problem 1 (10 points)**.** In your blue book, fill in each player's payoffs in the game

	L	R
T		
B		

so that all of the following hold:

(a) (T, L) is a Nash equilibrium;

(b) for player 1, strategy T is weakly dominated by strategy B;

(c) for player 2, strategy L is weakly dominated by strategy R.

Problem 2 (15 points)**.** Consider the following two-period game between two players. In the first period, they play the classical prisoner's dilemma:

	C	D
C	$2, 2$	$-1, 3$
D	$3, -1$	$0, 0$

In the second period, after observing the action profile chosen in the first period, they play the following game:

	L	R
T	$2, 2$	$0, 0$
B	$0, 0$	$1, 1$

There is no discounting. Each player's payoff in the dynamic game is the average of her payoffs in the two periods.

1. How many pure strategies does each player have?

2. Find a (pure) subgame-perfect Nash equilibrium in which (C, C) is played in the first period.

Problem 3 (20 points)**.** This problem considers infinitely repeated Bertrand price competition between two firms, labeled $i = 1, 2$.

The stage game is standard Bertrand price competition. Simultaneously, each firm i chooses a price p_i in $[0, 1]$. The quantity demanded from firm 1 is

$$q_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 > p_2, \\ (1 - p_1)/2 & \text{if } p_1 = p_2, \\ 1 - p_1 & \text{if } p_1 < p_2. \end{cases}$$

The quantity demanded from firm 2 is symmetric. There are no production costs. Each firm's utility $u_i(p_1, p_2)$ equals its profits.

In the infinitely repeated game, the stage game is played in each period $t = 0, 1, 2, \ldots$. Past prices are perfectly observed. There is a common discount factor δ in $(0, 1)$. Hence, for $i = 1, 2$, firm i's utility is

$$U_i(p^0, p^1, \ldots) = \sum_{t=0}^{\infty} \delta^t u_i(p^t),$$

where p^t denotes the price profile chosen by the firms in period t.

1. In the stage game, find a symmetric price profile (p^*, p^*) that maximizes the sum of the firms' profits (over all price profiles).

2. Find a symmetric Nash equilibrium of the stage game. Denote this price profile by $(p^{\text{NE}}, p^{\text{NE}})$.

3. In the repeated game, consider the following grim-trigger strategy profile. Initially, each firm chooses price p^*. Thereafter, each firm chooses price p^* unless either firm has deviated, in which case it chooses price p^{NE} forever after. For which discount factors δ is this strategy profile a subgame perfect Nash equilibrium?

Problem 4 (15 points)**.** Consider the following Bayesian game. There are two players, denoted $i = 1, 2$. Each player simultaneously chooses whether to invest (I) or not (N). Payoffs are given in the following matrix.

	I	N
I	θ_1, θ_2	$\theta_1 - 1, 0$
N	$0, \theta_2 - 1$	$0, 0$

Each player i privately knows his payoff parameter θ_i in $[0, 1]$. The parameters θ_1 and θ_2 are drawn independently and uniformly from the unit interval $[0, 1]$.

1. Find three Bayesian Nash equilibria of this game.

2. Give intuition for why this game has multiple equilibria.

Problem 5 (20 points)**.** Consider an auction for a single good. There are two bidders, labeled $i = 1, 2$. Each bidder privately knows his own valuation for the good, which is uniformly distributed over $[0, 1]$, independent of the other bidder's valuation. The auctioneer runs a second-price auction. That is, each bidder is asked to submit a nonnegative bid. The highest bidder wins the good and must pay the second-highest bid. The loser pays nothing. (Ties are broken with a fair coin flip.)

1. Find a symmetric Bayesian Nash equilibrium of this auction.
2. Find an asymmetric Bayesian Nash equilibrium of this auction in which bidder 1 always wins the good. Explain in words why the strategy profile that you constructed is a Bayesian Nash equilibrium.
3. Is the auctioneer's revenue the same in parts 1 and 2? Is this consistent with the revenue equivalence theorem? Explain.

Problem 6 (15 points)**.** The local government is deciding whether to widen the bike lane on Memorial Drive. There are three agents, labeled $i = 1, 2, 3$, who would be affected by the widened bike lane. Agents 1 and 2 both bike, so they would benefit from a wider bike lane. Each agent $i = 1, 2$ privately knows her nonnegative valuation v_i from the bike lane being widened. Agent 3 drives to work and she finds a wider bike lane inconvenient. Agent 3 privately knows her nonnegative cost c_3 from the bike lane being widened.

For the bike lane project, the local government uses a VCG mechanism. At each of the reported profiles below, compute the VCG mechanism's allocation (whether the bike lane is widened) and transfers (how much each agent must pay to the government).[1]

1. $(v_1, v_2, c_3) = (2, 2, 1)$.
2. $(v_1, v_2, c_3) = (2, 3, 4)$.
3. $(v_1, v_2, c_3) = (2, 2, 5)$.

[1]Here, VCG refers to the version of the Vickrey–Clarke–Groves mechanism in which each agent "pays his externality."

Problem 7 (25 points)**.** Consider a seller and a buyer. The seller has a single good, which is either of high (H) or low (L) quality. The good is high quality with probability π, where $0 < \pi < 1$. If the quality of the good is $j \in \{L, H\}$ and the good is sold at price p, then the utilities for the seller and the buyer are given by

$$u_S = p - c_j, \qquad u_B = v_j - p.$$

If the good is not sold, then both players get zero utility.

Assume that

$$0 < c_L < c_H < v_L < v_H < 1.$$

All parameters of the game are common knowledge. The seller privately knows the quality of the good; the buyer does not.

For parts 1 and 2, consider the following protocol. The seller offers a price p in $[0, 1]$. The buyer observes the offer and chooses whether to accept or reject it. If the offer is rejected, then the good is not sold.

1. Given a price p^* satisfying $v_L \leq p^* \leq (1 - \pi)v_L + \pi v_H$, find a perfect Bayesian equilibrium in which the good is always sold at price p^*. (Remember to specify all components of the equilibrium and to explain your notation clearly.)

2. Explain why there cannot be a separating perfect Bayesian equilibrium in which the good is always sold.

Now suppose that the seller can obtain a *product certification*. The seller chooses both a price p in $[0, 1]$ and whether to certify the good. The buyer observes the good's price and whether it is certified, and then chooses to accept or reject the offer. If the offer is rejected, then the good is not sold.

High- and low-quality goods can both be certified, but the quality $j \in \{L, H\}$ affects the seller's cost κ_j of certification. Namely, if the good of quality j is certified and then sold at price p, then the utilities for the seller and the buyer are given by

$$u_S = p - c_j - \kappa_j, \qquad u_B = v_j - p,$$

where $0 < \kappa_H < \kappa_L$. If the good is not sold, then both players get zero utility (i.e., the seller does not pay the certification cost if the good is not sold.)

3. For which prices p_L and p_H does there exist a separating perfect Bayesian equilibrium in which: (a) the low-quality good is offered *without certification* at price p_L, (b) the high-quality good is offered *with certification* at price p_H, and (c) the good is always sold?

4. Since both high- and low-quality goods can be certified, certification may seem useless. What purpose does certification serve in the equilibrium in part 3?