Fall 2025
14.12 Game Theory
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Problem Set 1

Problem 1 (A decision problem under uncertainty)**.** A student is taking a pass–fail class, and she is deciding whether to study for the exam. The student believes that she will pass with probability p_S if she studies and she will pass with probability p_N if she does not study, where $0 < p_N < p_S < 1$.

1. Formulate this problem in the abstract framework from class. That is, define the set Z of consequences/outcomes and specify a von Neumann–Morgenstern (vNM) utility function $u\colon Z \to \mathbf{R}$. Your utility function should capture the idea that the student likes passing and doesn't like studying (but there are many correct answers).

2. Which lottery is induced by studying? Which lottery is induced by not studying? Compute the expected utility from each choice.

3. For which values of p_N and p_S does each of the following hold:

   (a) the student strictly prefers studying;

   (b) the student strictly prefers not studying;

   (c) the student is indifferent between studying and not studying.

   Your answer will depend on your specification of u in part 1.

4. The student discusses this problem with a friend, and he reasons as follows. "You don't know for sure whether you will pass. But you should formulate a belief about the probability that you will pass. Whatever that passing probability is,

you would prefer to get that pass–fail lottery and not study than to get that lottery and study. So you shouldn't study." What is the error in this reasoning?

Problem 2 (von Neumann-Morgenstern (vNM) utility representation)**.** The vNM utility function is *cardinal*, so the exact values matter, not just their order. Nevertheless, there are generally multiple vNM representations of the same preferences. Let Z be a finite set of consequences. Consider two vNM utility functions $u_1, u_2 \colon Z \to \mathbf{R}$. Let $U_1, U_2 \colon \Delta(Z) \to \mathbf{R}$ be the associated expected utility functions. (Recall that $\Delta(Z)$ denotes the set of lotteries over Z.) Suppose that $u_2 = \alpha u_1 + \beta$, for some constants $\alpha > 0$ and $\beta \in \mathbf{R}$. Show that u_1 and u_2 represent the same preferences over lotteries. That is, for all lotteries $p, q \in \Delta(Z)$, we have

$$U_1(p) \geq U_1(q) \iff U_2(p) \geq U_2(q).$$

[The symbol $\iff$ means *if and only if*, or equivalently, *implies and is implied by*.] Does this result hold if $\alpha < 0$? What if $\alpha = 0$?

Problem 3 (Guessing someone else's coin)**.** Alice and Bob are held in separate prison cells. The warden proposes the following game between Alice and Bob. First, the warden will go to Alice's cell and flip a fair coin so that Alice sees the result. Alice is then asked to guess the outcome of a coin flip that the warden is about to perform in Bob's cell. Next, the warden goes to Bob's cell and flips a fair coin so that Bob sees the result. Bob is then asked to guess the outcome of the coin flip that the warden performed in Alice's cell.

Both prisoners will be released if at least one correctly guesses the outcome of the coin flip performed in the other's cell. Otherwise, neither prisoner will be released. Each prisoner gets utility 1 from being released and utility 0 otherwise.

(a) Write this as an extensive-form game.

(b) Write this as a strategic-form game.

(c) Which strategy profiles result in the highest probability of release? Which strategy profiles result in the lowest probability of release? Do you find this result surprising?

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