Fall 2025
14.12 Game Theory
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Problem Set 10

Problem 1 (VCG mechanism)**.** There are two goods, α and β, and two bidders, $i = 1, 2$. The goods will be auctioned to the bidders according to a VCG mechanism. There are four possible (pure) allocations of the goods: good α can be allocated to player 1 or player 2; good β can be allocated to player 1 or player 2.[1] The principal can also allocate the goods randomly, e.g., by flipping a c oin. Formally, a (random) *allocation* is a probability distribution over the four pure allocations of the two goods among the two players.

Each bidder i has one of three possible valuations, denoted v^A, v^B, v^C. Each player's utility depends on the set of goods allocated to him, as indicated in the table below:

	v^A	v^B	v^C
$\varnothing$	0	0	0
$\{\alpha\}$	4	2	3
$\{\beta\}$	2	4	3
$\{\alpha, \beta\}$	5	5	6.5

Compute the VCG mechanism,[2] breaking ties symmetrically between players $i = 1$ and $i = 2$. That is, for all $\hat{v}_1, \hat{v}_2 \in \{v^A, v^B, v^C\}$ compute an efficient (random) allocation $x^*(\hat{v}_1, \hat{v}_2)$ and the associated VCG transfers $t_1(\hat{v}_1, \hat{v}_2)$ and $t_2(\hat{v}_1, \hat{v}_2)$.

[1] Here, we are prohibiting the auctioneer from keeping either good. Allowing that possibility would not change the VCG mechanism.

[2] Specifically, each player pays her "externality."

Problem 2 (Perfect Bayesian Equilibrium)**.** Exercise 15.12.
Hint for part b: If Nature's move is represented explicitly in the extensive form, then Bob will have one information set, which contains four nodes.

Problem 3 (Job-market signaling)**.** In class, we analyzed a simple job-market signaling game in which the sender (worker) had two possible types (abilities), ℓ and h. This problem considers a generalization of that setting.

The sender has three possible types $t \in \{\ell, m, h\}$, where $0 < \ell < m < h$. The sender's type is drawn from a prior distribution that puts probabilities π_ℓ, π_m, and π_h on types ℓ, m, and h, respectively. The sender knows her type and chooses an education level $e \in [0, 1]$. The receiver (firm) sees the sender's education level, but not the sender's type, and then chooses a wage $w \geq 0$. The payoffs for the sender and receiver are

$$u_S(e, w; t) = w - \frac{1}{t}e^2, \qquad u_R(e, w; t) = -(w - t)^2.$$

1. For which education levels e^* does there exist a *pooling* PBE[3] in which every sender type chooses education e^*?

2. For which distinct education levels $\underline{e}$ and $\bar{e}$ does there exist a *semi-separating* PBE in which type ℓ chooses $\underline{e}$ and types m and h choose $\bar{e}$?

3. For which distinct education levels $\underline{e}'$ and $\bar{e}'$ does there exist a *semi-separating* PBE in which types ℓ and m choose $\underline{e}'$ and type h chooses $\bar{e}'$?

4. For which distinct education levels e_ℓ, e_m, and e_h does there exist a *separating* equilibrium in which each type t chooses e_t, for $t = \ell, m, h$?

Problem 4 (Price signaling)**.** Consider the incomplete-information bargaining setting from class. There are two players: a seller and a buyer. The seller has a single good, which is either of high (H) or low (L) quality. The good is high quality with probability π, where $0 < \pi < 1$. If the quality of the good is $j \in \{L, H\}$ and the good is exchanged at price p, utilities for the seller and the buyer are given by

$$u_S = p - c_j, \qquad u_B = v_j - p.$$

[3]Here and below, PBE means *perfect Bayesian equilibrium.*

If there is no exchange, then both players get zero utility.

We study the signaling setting in which the seller, but not the buyer, observes the quality of the good. The seller chooses a price offer p at which to sell the good. The buyer can either accept this offer or else reject the offer, in which case there is no exchange.

Assume that $c_L < c_H$; $v_L < v_H$; $c_L < v_L$; and $c_H < v_H$. All these parameters are commonly known (but the buyer does not observe the realized quality of the good).

1. Give a necessary and sufficient condition on the parameters π, c_L, c_H, v_L, v_H for there to exist a pooling equilibrium in which the good is always exchanged.

2. Show that there is no separating equilibrium in which the good is always exchanged.

3. Find a separating equilibrium with prices $p_L < p_H$ in which the buyer buys with positive probability when facing price p_H.

4. Find a semi-separating equilibrium with prices $p_1 < p_2$ such that the seller offers prices p_1 and p_2 with positive probability when quality is low, and always offers price p_2 when quality is high.