Fall 2025
14.12 Game Theory
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Problem Set 4

Problem 1 (Chess, zero-sum games, and backward induction)**.** Chess is a finite extensive-form game of perfect information.[1] Suppose that each player gets a payoff of 1 if she wins, 0 if she draws, and -1 if she loses. With these payoffs, chess (in strategic form) is a zero-sum game.

(a) Appealing to backward induction and the minimax theorem, argue that the value of chess must be either 1, 0, or -1. (The value of chess is not currently known.)

(b) This part illustrates why part (a) required an argument. Give an example of a zero-sum game in which player 1's payoffs (from pure strategy profiles) take values only in $\{1, 0, -1\}$,[2] but the value of the game is not 1, 0, or -1.

(c) Now consider a variant of chess. Player 1 (white) can initially make a single move or two consecutive legal moves. Then Player 2 (black) moves once, Player 1 moves once, Player 2 moves once, and so on as in usual chess. Argue that the value of this chess variant must be either 1 or 0. *To answer this question, you only need to know (or look up) the basic moves that each piece can make.*

Problem 2 (BoS with observed moves and a switching cost)**.** This problem considers a variant of the BoS game from class. Alice (player 1) and Bob (player 2) are each choosing between the Celtics game (C) and the Red Sox game (S). As in class, the

[1] Chess is finite due to the rule that the game is declared a draw if the same board position is reached three times.

[2] Player 1's payoffs need *not* take all three values 1, 0, and -1.

payoff profile is $(2,1)$ if both players go to the Celtics game; $(1,2)$ if they both go to the Red Sox game; and $(0,0)$ if they go to different games.

The timing is as follows. First Alice chooses which game to attend. Bob observes Alice's choice and then chooses which game to attend. Finally, Alice observes Bob's choice and then chooses either to stay where she is or "switch" by traveling to the other game. Suppose that Alice must pay a utility cost c if she switches after seeing Bob's choice.

(a) Apply backward induction to this game in each of the following cases: (i) $0 < c < 1$; (ii) $1 < c < 2$; (iii) $c > 2$.

(b) Discuss how the cost c affects the result of backward induction. Is Alice better off with a lower cost c? Interpret this finding.

Problem 3 (Inheritance)**.** Consider the following game, involving intergenerational conflicts in the presence of altruism. There are 3 players, labeled $1, 2, 3$.

- Player 1 starts with wealth level $w_1 > 0$ and chooses a consumption amount $c_1 \in [0, w_1]$ for herself, leaving $w_2 = w_1 - c_1$ to Player 2.
- Then Player 2 observes the consumption choice c_1, chooses her consumption level $c_2 \in [0, w_2]$ for herself, leaving $w_3 = w_2 - c_2$ for Player 3.
- Finally, Player 3 observes the consumption history (c_1, c_2) and then chooses consumption $c_3 \in [0, w_3]$.

The utility of each player i is

$$u_i = \sqrt{c_i} + \beta \sum_{j \neq i} \sqrt{c_j},$$

for some $\beta \in (0, 1)$.

(a) Interpret the parameter β.

(b) Apply backward induction to this game. What is the resulting consumption level for each player?

(c) Suppose that player 1 could unilaterally choose c_1, c_2, and c_3 subject to the constraint that $c_1 + c_2 + c_3 = w_1$. What would she choose? How do these consumption levels compare with those under the backward induction solution? Discuss.

Problem 4 (Pretrial negotiation)**.** Consider the following single-proposal protocol for pretrial negotiation between Plaintiff and Defendant. There are five commonly known parameters $J, \hat{c}_P, \hat{c}_D, c_P, c_D$, which are all strictly positive. The negotiation proceeds in three phases.

I. *Invitation to pretrial negotiation.* The Plaintiff chooses whether to invite the Defendant to pretrial negotiation. If the Defendant is invited, he can accept or reject the invitation. If the invitation is made and accepted, play proceeds to stage II, which costs c_P for the Plaintiff and c_D for the Defendant. Otherwise, the game proceeds to stage III.

II. *Pretrial negotiation.* The Proposer (to be specified below) chooses a settlement $s \in \mathbf{R}$ to offer to the Respondent (the other player). The Respondent chooses to accept or reject this proposed settlement. If the settlement s is accepted, then the Defendant pays s to the Plaintiff and the game ends. Otherwise, play proceeds to stage III.

III. *Trial.* The Plaintiff chooses whether to bring the suit to trial. If she does not bring the suit to trial, the game ends. If she brings the suit to trial, then the Plaintiff and the Defendant pay trial costs $\hat{c}_P$ and $\hat{c}_D$, respectively. At the end of the trial, the Defendant pays J to the Plaintiff.

As in class, there is no discounting and both players are risk-neutral. For this protocol, answer the following questions.

(a) Assuming $\hat{c}_P > J$, apply backward induction to stage III. (That is, for this part only, you may assume that the game consists only of Stage III.)

For the remaining parts, assume $\hat{c}_P < J$.

(b) Suppose the Plaintiff is the Proposer. Write out the extensive-form game and apply backward induction.

(c) Suppose the Defendant is the Proposer. Write out the extensive-form game and apply backward induction.

(d) Now suppose that at the beginning of stage II, a fair coin is flipped to determine which party is the Proposer. Write out the extensive-form game. For which cost parameters is there a backward induction solution in which the case is settled out of court (with certainty)?

(e) Finally, suppose the coin can be biased so that the Plaintiff is the Proposer with commonly known probability $\alpha \in (0, 1)$. We want to find a probability α such that there is a backward induction solution in which the case is settled out of court (with certainty). For which cost parameters does such an α exist? Interpret your condition on the cost parameters.