Fall 2025
14.12 Game Theory
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Problem Set 9

Note: Problem 4 is about *ad auctions*, which will be covered in lecture on Tuesday, November 18.

Problem 1 (Another equilibrium of the second-price auction)**.** Suppose that there are two bidders, whose valuations are independently and uniformly distributed over $[0, 1]$. In class, we showed that the second-price auction has a Bayesian Nash equilibrium in which each bidder bids his valuation.

(a) Construct a different equilibrium of the second-price auction. (Hint: this equilibrium will be asymmetric; think about what happens if one bidder bids very low and the other bids very high.)

(b) Compute the auctioneer's expected revenue under the original "truthful" equilibrium and under this new equilibrium. Does your finding contradict the revenue formula from class?

Problem 2 (Reserve prices)**.** Suppose that there are two bidders, whose valuations are distributed independently and uniformly over $[0, 1]$. Recall from the last problem set (PS 8, Problem 4) the first-price auction with reserve price r, where $0 < r < 1$.

(a) Under the symmetric equilibrium you computed in that problem set, compute the expected payment $\hat{T}_i(v_i)$ made by bidder i when her valuation is v_i, and compute the probability $\hat{Q}_i(v_i)$ that bidder i wins the good when her valuation is v_i.

(b) The transfer function $\hat{T}_i$ is discontinuous at $v_i = r$, so we cannot directly apply the revenue formula from class.[1] Check that the following modification of the formula holds: for all $v_i \geq r$.

$$\hat{T}_i(v_i) = \hat{T}_i(r) + \int_r^{v_i} x\hat{Q}_i'(x)\,\mathrm{d}x.$$

Problem 3 (Failure of revenue equivalence with discrete valuations)**.** This problem shows that revenue equivalence can fail if the bidders' valuations are not drawn from an interval.

There are two players, denoted $i = 1, 2$. Each player i's valuation v_i takes two possible values, $\underline{v}$ and $\bar{v}$, each with probability $1/2$ independently of the other player's valuation. Suppose $0 < \underline{v} < \bar{v} < 1$.

The auctioneer uses a *posted price mechanism* with a fixed price p in $(\underline{v}, \bar{v})$. Each player i says whether she would like to buy at price p, denoted Y for *yes* or N for *no*. If no one chooses Y, then nothing happens. If one player chooses Y, then that player gets the good for price p. If both players choose Y, then the auctioneer flips a fair coin to determine whom to sell to (at price p).

Find a Bayes–Nash equilibrium of this posted-price mechanism. Under this equilibrium, when bidder i has valuation v_i, compute bidder i's win probability $\hat{Q}_i(v_i)$ and expected payment $\hat{T}_i(v_i)$. Show that changing the value of p can change the expected payment functions $\hat{T}_i$, without changing $\hat{T}_i(\underline{v})$ or $\hat{Q}_i$.

Problem 4 (Ad auction)**.** Consider a special case of the ad auction setting from class. There are three ad slots, with click-through rates $\alpha_1 > \alpha_2 > \alpha_3$. The auction slots are allocated using a VCG mechanism. This problem illustrates why bidding truthfully is optimal in such an auction. Suppose that you are one of three bidders in the auction. Suppose further that your value-per-click is v and that your two opponents bid b_1 and b_2, where $b_1 > b_2$.[2] Compute your expected utility from bidding b, in each of the following three cases:

i. $b > b_1 > b_2$;

[1] Our proof assumed that the function $\hat{T}_i$ is differentiable. Using a different proof, it can be shown that a version of the revenue equivalence theorem holds without this assumption.

[2] This is a thought experiment. In reality, you wouldn't know your opponents' bids.

ii. $b_1 > b > b_2$;

iii. $b_1 > b_2 > b$.

Denote these three expected utilities by U_1, U_2, U_3, respectively. Show that U_1 is highest if $v > b_1$; U_2 is highest if $b_1 > v > b_2$; and U_3 is highest if $b_2 > v$.