Global Games

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14.126 Game Theory

Motivation

- Outcomes may differ in similar environments.
- This has been explained by multiple equilibria (w/strategic complementarity)
  - Investment/Development
  - Search
  - Bank Runs
  - Currency attacks
  - Electoral competition…
- But with introduction of incomplete information, such games tend to be dominance-solvable

A simple partnership game

	Invest	Not-Invest
Invest	θ,θ	$\theta-1, 0$
Not-Invest	$0,\theta-1$	$0,0$

- Assume θ is known
- If $\theta > 1$, Invest is dominant
- If $\theta < 0$, Not-Invest is dominant
- Otherwise, multiple equilibria
- Risk-dominance v. Pareto-dominance

θ is not common knowledge

- θ is uniformly distributed over a large interval
- Each player *i* gets a signal

$$x_i = \theta + \sigma\varepsilon_i$$

  - $(\varepsilon_1, \varepsilon_2)$ is bounded,
  - Independent of θ,
  - and has joint density.
- Carlsson and van Damme: when is σ small, the game is dominance solvable:
  - Invest if $x_i > ½$
  - Not Invest if $x_i < ½$.

Motivation—Literature Review

- Carlsson and van Damme ‘93 shows this more generally for 2 x 2 games
- The unique solution is given by risk-dominance (≅ best response to uniform belief)
- Morris and Shin ‘98 applies this idea to “currency attack” problem, obtaining intuitive comparative statics,
- … and leading to a large applied literature

Road map

1. Carlsson and Van Damme, briefly
2. Global Games as Supermodular Games
   1. 2x2 Example
   2. Frankel Pauzner and Morris
3. Currency Attacks & Applications
4. Dynamic Global Games

Carlsson and Van Damme—2x2 games

RISK-DOMINANCE

p-dominance

- Consider a game $G = (N, S, u)$
- A Nash equilibrium $s^* = (s_1^*, \dots, s_n^*)$ is $(p_1, \dots, p_n)$-dominant if, for each i, s_i^* is a best response whenever i assigns at least probability p_i on s_{-i}^*.
- In a 2x2 game an equilibrium is risk dominant if it is (p_1, p_2)-dominant for some
$$p_1 + p_2 < 1.$$
- In a symmetric 2x2 game, risk dominance = best response to uniform belief

Risk-Dominance

	a	b
a	$g(a,a)$	$g(a,b)$
b	$g(b,a)$	$g(b,b)$

- Assume (a,a) and (b,b) are Nash equilibria.
- For each equilibrium s and player i, let
- (a,a) is risk dominant iff $\left(g_1(a,a) - g_1(b,a)\right)$

Monotone supermodular games

- $G = (N,T,A,u,p)$
- $T = T_0 \times T_1 \times \ldots \times T_n \subseteq \mathrm{R}^M$ [set of type profiles]
- A_i is compact sublattice of R^K [set of actions]
- $u_i : A \times T \to \mathrm{R}$ [utility function]
  - $u_i(a,.)$: $T \to \mathrm{R}$ is measurable
  - $u_i(.\ ,t)$: $A \to \mathrm{R}$ is continuous, "bounded", supermodular in a_i, has increasing differences in a and in (a_i,t)
- $p(.|t_i)$ is increasing function of t_i—in the sense of 1st-order stochastic dominance (e.g. p is affiliated). [interim belief]
- Theorem: There exist BNE s^* and s^{**} such that
  - For each rationalizable strategy s, $s^* \geq s \geq s^{**}$;
  - Both s^* and s^{**} are isotone.

A simple partnership game

	Invest	Not-Invest
Invest	θ,θ	$\theta-1, 0$
Not-Invest	$0,\theta-1$	$0,0$

- θ is uniformly distributed over a large interval
- Each player i gets a signal

  $$x_i = \theta + \sigma\varepsilon_i$$

  where $(\theta,\varepsilon_1, \varepsilon_2)$ is bounded and stochastically independent.

Monotone BNE

- Best reply:

  $$\text{Invest} \Leftrightarrow x_i \geq \Pr(s_j = \text{Not-Invest}|x_i)$$

- Assume supp(θ) = $[a,b]$ where $a < 0 < 1 < b$.
- $x_i < 0 \Rightarrow s_i(x_i)$ = Not Invest
- $x_i > 1 \Rightarrow s_i(x_i)$ = Invest
- A cutoff x_i^* s.t.
  - $x_i < x_i^* \Rightarrow s_i(x_i)$ = Not Invest; $x_i > x_i^* \Rightarrow s_i(x_i)$ = Invest;
- Symmetry: $x_1^* = x_2^* = x^*$
- $x^* = \Pr(s_j = \text{Not-Invest}|x^*) = \Pr(x_j < x^*|x_i = x^*) = 1/2$
- “Unique” BNE, i.e., “dominance-solvable”

Rank Beliefs

- Rank Belief:

$$R(x) = \Pr(x_j \leq x | x_i = x)$$

- Extremal Equilibria: monotone, symmetric BNE with cutoff x^*.
- Extremal equilibria are the extremal solutions to the Indifference Condition for Cutoff:

$$R(x^*) = E[\theta | x_i = x^*]$$

Normal Model

- $\theta = y + \tau\eta$ and $x_i = \theta + \sigma\varepsilon_i$ where $\eta, \varepsilon_i \sim N(0,1)$ iid
- Conditional on x_i, with $\alpha = \tau^2/(\tau^2+\sigma^2)$, we have

$$E[\theta|x_i] = y + \alpha(x_i - y)$$
$$\theta \sim N(E[\theta|x_i], \sigma^2\alpha)$$
$$x_j \sim N(E[\theta|x_i], \sigma^2(\alpha + 1))$$
$$\varepsilon_i \sim N((1-\alpha)(x_i - y)/\sigma, \alpha)$$

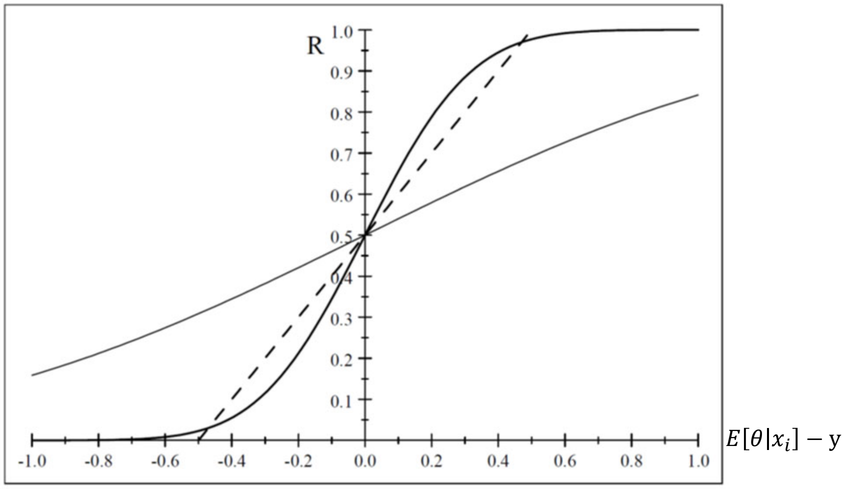
- Rank Belief Function:

$$R(x) = \Pr\left(x_j \leq x \middle| x_i = x\right) = \Phi(\lambda(E[\theta|x_i] - y))$$
$$\lambda = \frac{1-\alpha}{\alpha\sigma\sqrt{\alpha+1}} = \frac{\sigma}{\tau^2\sqrt{\alpha+1}}$$

- Indifference Condition for cutoff x*:

$$\Phi(\lambda(E[\theta|x_i] - y)) = E[\theta|x_i]$$

Equilibrium Cutoffs in Normal Model



Dominance Solvable if $\lambda < \sqrt{2\pi}$

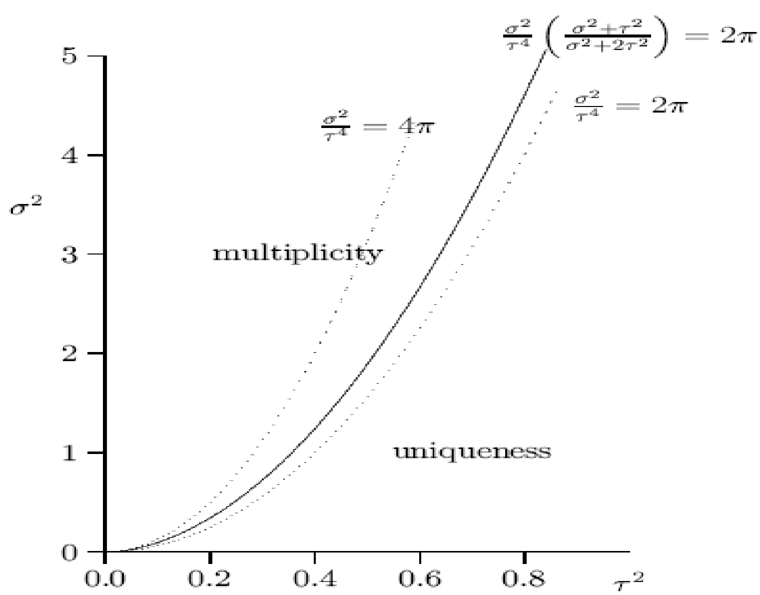


Figure 3.1: Parameter Range for Unique Equilibrium

Equilibrium Cutoffs

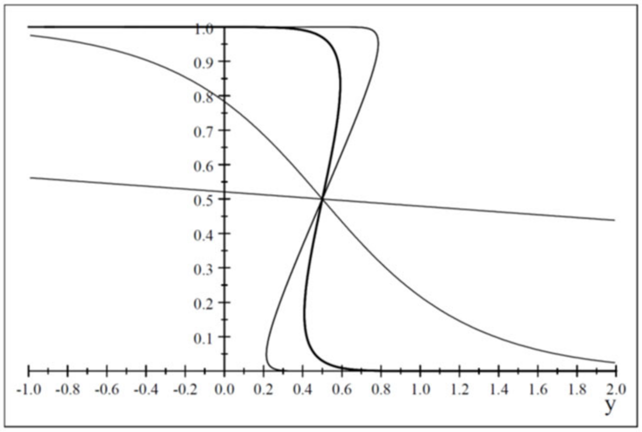
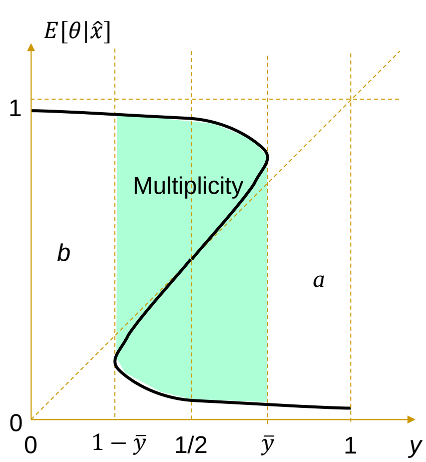
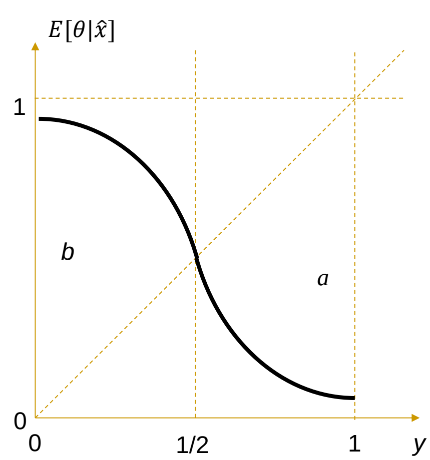


Figure 6.1: The equilibrium cutoff $E[\theta|\hat{x}]$ in Normal example as a function of y for $\lambda = 0.1, 1, 4, 10$.

Equilibrium Qutoffs



Multiplicity Region



Uniqueness Region

Equilibrium Selection

- $\Phi(\lambda(E[\theta|\hat{x}] - y)) = E[\theta|\hat{x}]$
- Carlsson and van Damme: $\sigma \cong 0$ while $\tau > 0$ is fixed ($\alpha \cong 1$, $\lambda \cong 0$)

$$E[\theta|\hat{x}] \cong \Phi(0) = 1/2$$

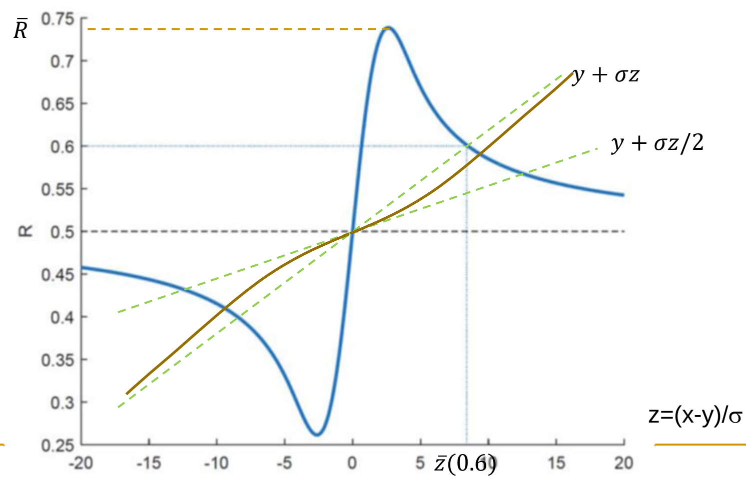
- i.e. risk dominant selection
- Alternatively, take $\lambda = 1$ (while $\sigma, \tau > 0$ can be arbitrarily small):

$$E[\theta|\hat{x}] = \Phi(E[\theta|\hat{x}] - y)$$

- Cutoff $E[\theta|\hat{x}]$ can take any value in (0,1), depending on y.
- i.e. **any** equilibrium can be selected by varying y.

Coordination under Model Uncertainty

- η has t-distribution (normal with unknown variance).
- $\sigma = \tau$



Coordination under Model Uncertainty

- Define
$$\bar{z}(\theta) = \max_z R^{-1}(\theta) = \max\{z | R(\sigma z + y) = \theta\}$$
- Invest is uniquely rationalizable if

1. it is risk dominant (i.e. $E[\theta|x_i] > 1/2$) and
2. there is large positive shock, i.e.,
$$E[\theta|x_i] - \mathrm{y} > \sigma\bar{z}(E[\theta|x_i]).$$

- In particular, invest is uniquely rationalizable whenever $E[\theta|x_i] > \bar{R}$.
- A converse is also true.
- Opposite conclusions without model uncertainty

General Supermodular Global Games Frankel, Morris, and Pauzner

Model

- $N = \{1,\ldots,n\}$ players
- $A_i \subseteq [0,1]$,
  - countable union of closed intervals
  - $0,1 \in A_i$
- Uncertain payoffs $u_i(a_i,a_{-i},\theta)$
  - continuous with bounded derivatives
- 1-dimensional payoff uncertainty: $\theta \in R$
- Each player *i* observes a signal

$$x_i = \theta + \sigma\varepsilon_i$$

  - $(\theta, \varepsilon_1, \varepsilon_2)$ are independent with atomless densities
  - $(\varepsilon_1, \varepsilon_2)$ bounded

Main Assumptions

Let $\Delta u_i(a_i,a'_i,a_{-i},\theta) = u_i(a_i,a_{-i},\theta) - u_i(a'_i,a_{-i},\theta)$

- Strategic complementarities: $a_i \geq a'_i$ & $a_{-i} \geq a'_{-i}$ $\Rightarrow \Delta u_i(a_i,a'_i,a_{-i},\theta) \geq \Delta u_i(a_i,a'_i,a'_{-i},\theta)$
- Dominance regions:
  - 0 is dominant when θ is very small
  - 1 is dominant when θ is very large
- State monotonicity: outside dominance regions, $\exists K>0$: $\forall\ a_i \geq a'_i\ \forall\ \theta \geq \theta'$,

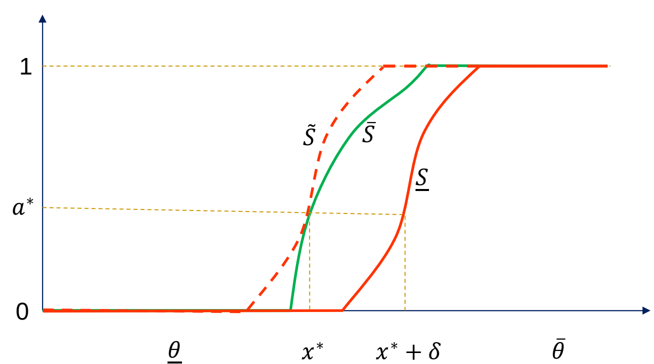
$$\Delta u_i(a_i,a'_i,a_{-i},\theta) - \Delta u_i(a_i,a'_i,a_{-i},\theta') \geq K(a_i - a'_i)(\theta - \theta')$$

Theorem (Limit Uniqueness)

- In the limit $\sigma \to 0$, there is a "unique" rationalizable strategy, which is increasing.
- i.e., there exists an increasing pure strategy profile s^* such that if for each $\sigma > 0$, s^σ is rationalizable at σ, then almost everywhere

$$\text{Lim}_{\sigma\to 0}\, s_i^\sigma(x_i) = s_i^*(x_i).$$

Intuition



Limit Solution

- $(s_1^*(x), s_2^*(x))$ is a Nash equilibrium of the complete information game in which it is common knowledge that $\theta = x$.

Noise dependence

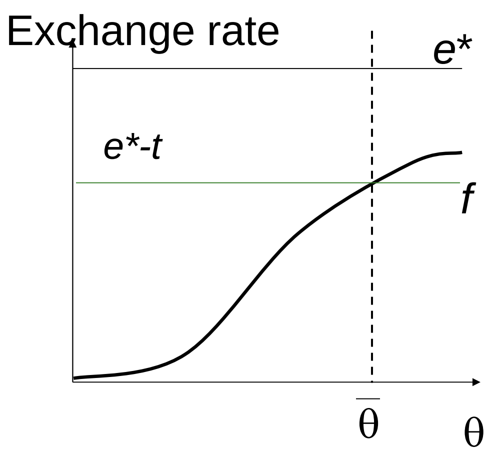
- There exists a game satisfying the FPM assumptions in which for different noise distributions, different equilibria are selected in the limit as the signal errors vanish.
- There are conditions under which s* is independent of the noise distributions.

Currency attacks
Morris & Shin

Model

- Fundamental: θ in [0,1]
- Competitive exchange rate: $f(\theta)$
- f is increasing
- Exchange rate is pegged at $e^* \geq f(1)$.
- A continuum of speculators, who either
  - Attack, which costs t, or
  - Not attack
- Government defends or not
- The exchange rate is e^* if defended, $f(\theta)$ otherwise

Speculator's Payoffs



- Attack, not defended:

  $e^* – f(\theta) – t$
- Attack, defended:

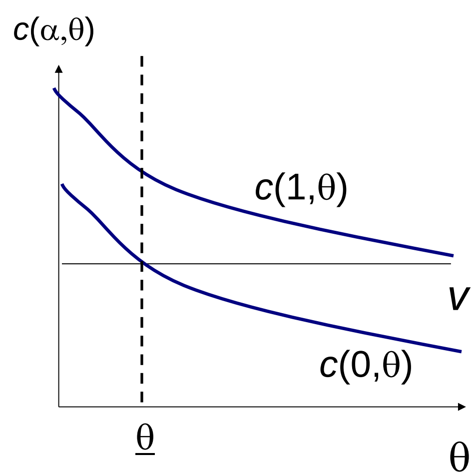
  $-t$
- No attack: 0

Government's payoffs

- Value of peg = v
- Cost of defending

  $c(\alpha,\theta)$

  where α is the ratio of speculators who attack
- c is increasing in α, decreasing in θ



Government's strategy

- Government knows α and θ;
- Defends the peg if

$$v > c(\alpha,\theta)$$

- Abandons it otherwise.

Information Structure

- θ is uniformly distributed on [0,1].
- Each speculator i gets a signal

$$x_i = \theta + \eta_i$$

- η_i's are independently and uniformly distributed on $[-\varepsilon, \varepsilon]$ where $\varepsilon > 0$ is very small.
- The distribution is common knowledge.

Government’s strategy

- Government knows α and θ;
- Defends the peg if

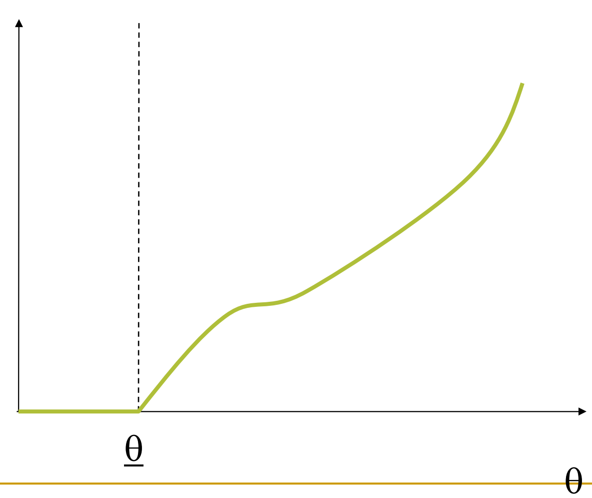
$$v > c(\alpha,\theta)$$

- Abandons it otherwise.

Define: $a(\theta)$ = the minimum α for which G abandons the peg

$$v = c(a(\theta),\theta)$$

$a(\theta)$



Speculator's payoffs

- *r* = ratio of speculators who attack
- u(Attack,*r*,θ) = *e** - *f*(θ) – *t* if *r* ≥ *a*(θ)

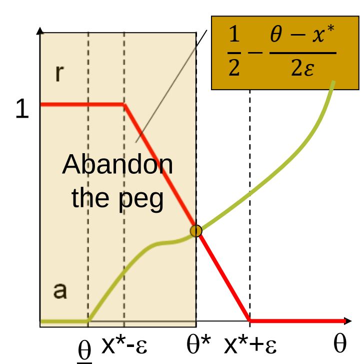
  -*t* otherwise
- U(NoAttack,*r*,θ) = 0

Unique Equilibrium

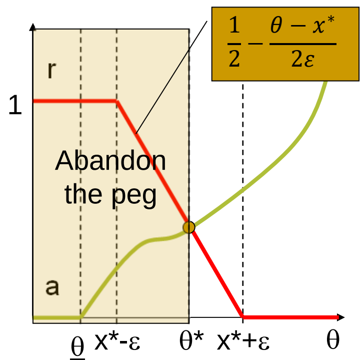
- **Equilibrium:** Attack iff $x_i \leq x^*$
- $r(\theta) = \Pr(x \leq x^* | \theta)$

$$a(\theta^*) = \frac{1}{2} - \frac{\theta^* - x^*}{2\varepsilon}$$

$$x^* - \varepsilon = \theta^* - 2\varepsilon(1 - a(\theta^*))$$



θ*



- Utility from attack

$$U(x^*) = \frac{1}{2\varepsilon}\int_{x^*-\varepsilon}^{\theta^*} (e^* - f(\theta))d\theta - t$$
$$\approx \frac{1}{2\varepsilon}(\theta^* - x^* + \varepsilon)(e^* - f(\theta^*)) - t$$
$$= (1 - a(\theta^*))(e^* - f(\theta^*)) - t$$
$$= 0$$

- Indifference Condition:

$$(1 - a(\theta^*))(e^* - f(\theta^*)) = t$$

"Risk dominance"

- Suppose all strategies are equally likely
- *r* is uniformly distributed on [0,1]
- Expected payoff from Attack

$$(1-a(\theta))(e^*-f(\theta)) - t$$

- Attack is "risk dominant" iff

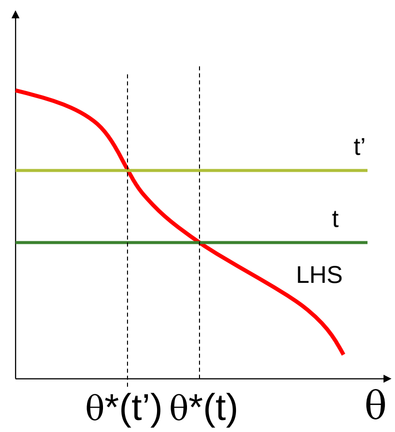
$$(1-a(\theta))(e^*-f(\theta)) > t$$

- Cutoff value θ*:

$$(1-a(\theta^*))(e^*-f(\theta^*)) = t$$

Comparative statics – t

- Cutoff value θ*:
  (1-a(θ*))(e*-f(θ*)) = t
- LHS is decreasing in θ*.

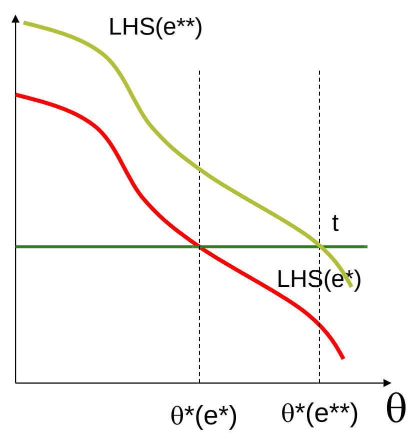


If transaction cost t increases,

attack becomes less likely!

Comparative statics – e*

- Cutoff value θ*:
  (1-a(θ*))(e*-f(θ*)) = t
- LHS is decreasing in θ*
- and increasing in e*



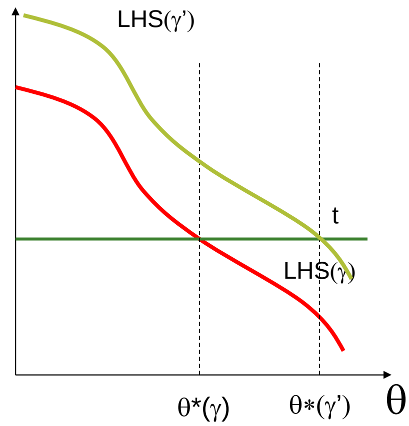
If e* increases,

attack becomes more likely!

Comparative statics – c

- Let c(α,θ) = γ C(α,θ)
- Cutoff value θ*:

  (1-a(θ*))(e*-f(θ*)) = t
- LHS is decreasing in θ*
- and decreasing in a
- i.e., increasing in γ



If γ increases, attack becomes more likely!

Continuum of Anonymous Players

- $N = [0,1]$
- $A_i = \{0,1\}$
- $\alpha(a) = \int a_j dj$
- Types: $x_i = \theta + \sigma\varepsilon_i$
  - θ distributed with CDF G, continuous pdf g
  - $\varepsilon_i \in [-1,1]$ distributed with CDF F and pdf f
- Payoff:

$$a_i U(\alpha, \theta)$$

  where U is weakly increasing
- 1 is dominant for $\theta > \overline{\theta}$ and 0 is dominant for $\theta < \overline{\theta}$

Extremal Equilibria

- Extremal equilibrium with cutoff $\hat{x}$
- Fraction of players who take action:

$$\alpha(\theta) = 1 - F\left(\frac{\hat{x} - \theta}{\sigma}\right) = F\left(\frac{\theta - \hat{x}}{\sigma}\right)$$

- Indifference Condition for cutoff:

$$\int_{\hat{x}-\sigma}^{\hat{x}+\sigma} U\left(F\left(\frac{\theta - \hat{x}}{\sigma}\right), \theta\right) dG(\theta|\hat{x}) = 0$$

- Linear Games:

$$U(\alpha, \theta) = \alpha + \theta - 1$$

- Indifference condition for linear games:

$$R(\hat{x}) = E[\theta|\hat{x}]$$

where

$$R(x) = \Pr\left(x_j \leq x \middle| x_i = x\right) = \int F(\varepsilon_i) dF(\varepsilon_i|x)$$

Games of Regime Change

- Payoffs:

$$U(\alpha, \theta) = \begin{cases} V(\theta) - C(\theta) & if \alpha \geq \overline{\alpha}(\theta) \\ -C(\theta) & if \alpha < \overline{\alpha}(\theta) \end{cases}$$

  - $V, \overline{\alpha}, C$ are Lipschitz continuous,
  - $V > 0$ weakly increasing,
  - $\overline{\alpha}, C$ are weakly decreasing
  - 0 is dominant if $\theta < \underline{\theta}$; 1 is dominant if $\theta > \overline{\theta}$
- Extremal Equilibria with Cutoff $\hat{x}$
- Regime Change if $\theta \geq \hat{\theta}$ where

$$\overline{\alpha}(\hat{\theta}) = F\left(\frac{\hat{\theta} - \hat{x}}{\sigma}\right)$$

- Indifference Condition:

$$\int_{\hat{\theta}}^{\infty} V(\theta) dG(\theta|\hat{x}) = E[C|\hat{x}]$$

- In the limit $\sigma \to 0$:

$$V(\hat{x})\left(1 - \overline{\alpha}(\hat{x})\right) = C(\hat{x})$$

Dynamic Global Games

- Selection by Dynamics
  - Burdzy, Frankel, Pauzner, Eca, 2001
  - Frankel, Pauzner, QJE, 2000
  - Chamley, QJE, 1999
- Dynamic Global Games of Regime Change
  - Angeletos, Hellwig, Pavan, Eca, 2007
- Fear of Miscoordination
  - Chassang, Eca, 2010
- Equilibrium Shifts and Large Shocks
  - Morris and Yildiz, 2019

Selection by Dynamics (BFP 01)

- A continuum of players, *i*
- Discrete time, t = 0, τ, 2τ, 3τ, …
- At each t, players randomly match to play the investment game where …
- Return θ_t follows a random walk ($\theta_t = \theta_{t-1} \pm \sigma\sqrt{\tau}$)
- Friction: Pr(*i* can change his action) = $k\tau$
- Players are forward looking
- Solution Concept: Iterated conditional dominance.
- **Theorem:** For small σ and large *k*, there exist a unique solution: risk dominant action everywhere

Dynamic Global Games of Regime Change (AHP,07)

- A continuum of players, i, discrete time, $t = 0, 1, 2, \dots$
- At each t, each player chooses {attack, no attack},
  - A_t= Fraction of people who attack
  - Regime changes if $A_t > \theta$
  - Payoff from attack $1 - c$ if regime changes – c otherwise
  - The game ends when the regime changes
- $\theta \sim N(z, 1/\alpha)$ and at each t,
- each i observes

$$x_{it} = \theta + \varepsilon_{it}$$

where $\varepsilon_{i0} \sim N(0{,}1/\beta)$; $\varepsilon_{it} \sim N(0{,}1/\eta_{it})$;

Dynamic Global Games of Regime Change

- First period play as in the static case:
  - Attack ⇔ $x_{i0} < \hat{x}$
  - Attack size: $A(\theta) = \Phi\left(\sqrt{\beta}(\hat{x} - \theta)\right)$
  - Critical threshold: $\hat{\theta} = \Phi\left(\sqrt{\beta}(\hat{x} - \hat{\theta})\right)$
  - Equilibrium Condition: $F(\hat{\theta}|\hat{x}) = c$
- A robust equilibrium for dynamic game:
  - no further attacks if regime survives $t = 0$
  - This is the only equilibrium when z is small
- Multiple monotone equilibria if z large;
  - Infinitely many equilibria with arbitrary # attacks if z is very large
- Interesting dynamics: periods of tranquility after supposed attacks

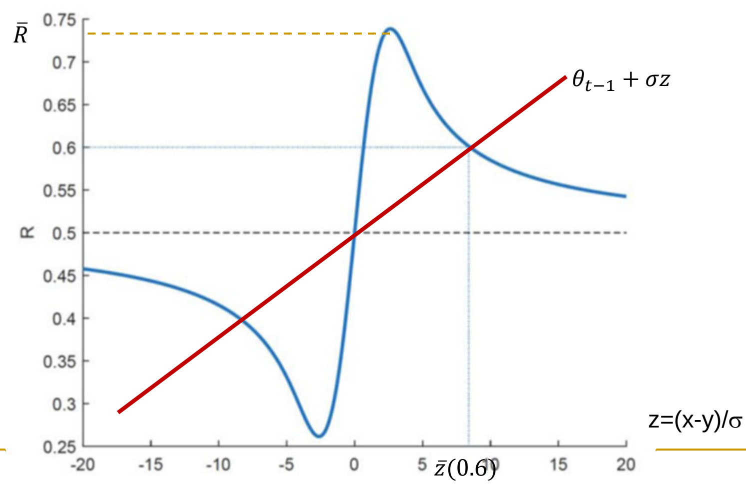
Equilibrium Shifts & Large Shocks (MY'19)

- Continuum of players i, discrete time t
- At each t, the following happens:
  - A new state θ_t is drawn:
    $$\theta_t = \theta_{t-1} + \sigma\eta_t$$
  - Each player i observes her own return parameter:
    $$x_t = \theta_t + \sigma\varepsilon_{it}$$
  - Each i chooses $a_{it} \in \{0,1 = invest\}$
  - Payoff from investment:
    $$x_t + A_t - 1$$

  where $A_t = \int a_{jt} dj$
- η_t has fat tails while ε_{it} has light tails

Equilibrium shifts & Large shocks

- Unique rationalizable action when there is a large shock or $x_{it} \notin (1-\bar{R}, \bar{R})$
- Multiple equilibrium actions otherwise



Equilibrium shifts and Large Shocks

A typical path under hysteresis