Implicit Differentiation (Rational Exponent Rule)

We know that if n is an integer then the derivative of $y = x^n$ is nx^{n-1}. Does this formula still work if n is not an integer? I.e. is it true that:

$$\frac{d}{dx}(x^a) = ax^{a-1}.$$

We proved this formula using the definition of the derivative and the binomial theorem for $a = 1, 2, ...$ From this, we also got the formula for $a = -1, -2, ...$ Now we'll extend this formula to cover rational numbers $a = \frac{m}{n}$ as well. In particular, this will let us take the derivative of $y = \sqrt[n]{x} = x^{1/n}$.

Suppose $y = x^{\overline{n}}$, where m and n are integers. We want to compute $\frac{dy}{dx}$. None of the rules we've learned so far seem helpful here, and if we use the definition of the derivative we'll get stuck trying to simplify $(x - \Delta x)^{m/n}$. We need a new idea.

The thing that's keeping us from using the definition of the derivative is that the denominator of n in the exponent forces us to take the n^{th} root of x. We could solve this problem by raising both sides of the equation to the n^{th} power:

$$\begin{aligned} y &= x^{\frac{m}{n}} \\ y^n &= (x^{\frac{m}{n}})^n \\ y^n &= x^{\frac{m}{n} \cdot n} \\ y^n &= x^m \end{aligned}$$

What happens if we try to take the derivative now by applying the operator $\frac{d}{dx}$? We have a rule for finding the derivative of a variable raised to an integer power; we can use this rule on both sides of the equation $y^n = x^m$.

$$\begin{aligned} y^n &= x^m \\ \frac{d}{dx} y^n &= \frac{d}{dx} x^m \end{aligned}$$

How do we compute $\frac{d}{dx} y^n$? We know that y is a function of x, so we can apply the chain rule with outside function y^n and inside function y. Suppose $u = y^n$. Then the chain rule tells us:

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx}$$

So

$$\frac{d}{dx} y^n = \left(\frac{d}{dy} y^n\right) \frac{dy}{dx} = ny^{n-1}\frac{dy}{dx}.$$

On the right hand side of the equation we have $\frac{d}{dx}x^m = mx^{m-1}$, so we end up with:

$$\begin{aligned}\frac{d}{dx}y^n &= \frac{d}{dx}x^m \\ ny^{n-1}\frac{dy}{dx} = mx^{m-1}\end{aligned}$$

We're left with only one unknown quantity in this equation — $\frac{dy}{dx}$ — which is exactly what we're trying to find. Can we solve for $\dfrac{dy}{dx}$ and use this to find the derivative of $y = x^{m/n}$? We can, but we need to use a lot of algebra to do it.

By dividing both sides by ny^{n-1} we get:

$$\frac{dy}{dx} = \frac{m}{n}\frac{x^{m-1}}{y^{n-1}}$$

This looks promising but we want our answer in terms of x, without any y's mixed in. To get rid of the y we can now substitute $x^{m/n}$ for y. (We couldn't have done this before taking the derivative because we don't know how to take the derivative of $x^{m/n}$ — that's the whole point!)

$$\begin{aligned}\frac{dy}{dx} &= \frac{m}{n}\left(\frac{x^{m-1}}{y^{n-1}}\right) \\ &= \frac{m}{n}\left(\frac{x^{m-1}}{(x^{m/n})^{(n-1)}}\right) \\ &= \frac{m}{n}\left(\frac{x^{m-1}}{x^{(m/n)\cdot(n-1)}}\right) \\ &= \frac{m}{n}\frac{x^{m-1}}{x^{m(n-1)/n}} \\ &= \frac{m}{n}x^{((m-1)-\frac{m(n-1)}{n})} \\ &= \frac{m}{n}x^{\frac{n(m-1)}{n}-\frac{m(n-1)}{n}} \\ &= \frac{m}{n}x^{\frac{n(m-1)-m(n-1)}{n}} \\ &= \frac{m}{n}x^{\frac{nm-n-nm+m}{n}} \\ &= \frac{m}{n}x^{\frac{m-n}{n}} \\ &= \frac{m}{n}x^{(\frac{m}{n}-\frac{n}{n})} \\ \text{So,}\quad \frac{dy}{dx} &= \frac{m}{n}x^{(\frac{m}{n}-1)}\end{aligned}$$

This is the answer we were hoping to get! We now know that for any rational number a, the derivative of x^a is ax^{a-1}.