

## The Mean Value Theorem and Linear Approximation

What's the difference between the mean value theorem and the linear approximation?

The linear approximation to  $f(x)$  near  $a$  has the formula:

$$f(x) \approx f(a) + f'(a)(x - a) \quad x \text{ near } a.$$

If we let  $\Delta x = x - a$ , we get:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x - a) \\ f(x) - f(a) &\approx f'(a)\Delta x \\ \frac{\Delta f}{\Delta x} &\approx f'(a). \end{aligned}$$

Similarly the MVT says:

$$f(b) = f(a) + f'(c)(b - a) \quad \text{for some } c, a < c < b$$

If  $b$  is near  $a$  then we can write  $b - a = \Delta x$  and rewrite the theorem as:

$$\frac{\Delta f}{\Delta x} = f'(c) \quad \text{for some } c, a < c < b.$$

The mean value theorem tells us that  $\frac{\Delta f}{\Delta x}$  is *exactly* equal to  $f'(c)$  for some  $c$  between  $a$  and  $b$ . We don't know precisely where  $c$  is; it depends on  $f$ ,  $a$ , and  $b$ .

As Professor Jerison says in the video, this is telling us that the average change on the interval is between the maximum and minimum values  $f'(x)$  reaches on the interval  $[a, b]$  (because the derivative is continuous).

$$\min_{a \leq x \leq b} f'(x) \leq \frac{f(b) - f(a)}{b - a} = f'(c) \leq \max_{a \leq x \leq b} f'(x)$$

In other words, the average speed of your trip is somewhere between your minimum speed and your maximum speed.

Linear approximation, is based on the assumption that the average speed is approximately equal to the initial (or possibly final) speed. Figure 1 illustrates the approximation  $1 + x \approx e^x$ .

If the interval  $[a, b]$  is short,  $f'(x)$  won't vary much between  $a$  and  $b$ ; the max and the min should be pretty close. The mean value theorem tells us absolutely that the slope of the secant line from  $(a, f(a))$  to  $(x, f(x))$  is no less than the minimum value and no more than the maximum value of  $f'$  on that interval, which assures us that the linear approximation does give us a reasonable approximation of the  $f$ .

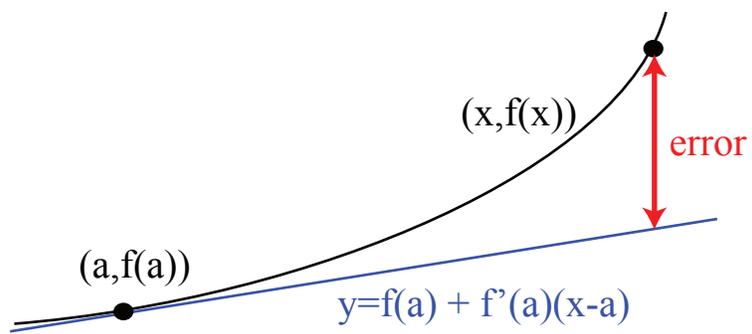


Figure 1: MVT vs. Linear Approximation.

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