Secants and Tangents

We defined the tangent line as a limit of secant lines. We also know that as Δx approaches 0 the secant's slope $\frac{\Delta f}{\Delta x}$ approaches the slope of the tangent line. How close to 0 does Δx have to be for $\frac{\Delta f}{\Delta x}$ to be close to the slope of the tangent line?

We'll use the Secant Approximation mathlet to look at a few examples. Use the dropdown menu in the lower left corner to select the function $f(x) = 0.5x^3 - x$. Use the red and yellow sliders to answer part (a) of each question, then use the Tangent checkbox to answer part (b). Be sure to uncheck Tangent before starting the next problem.

You may find it helps to work with a partner on this exercise.

1. Move the red slider to $x = -0.75$; we'll investigate the slopes of secant lines passing through the point $(-0.75, f(-0.75))$.

   (a) Use the yellow slider to find the value of $\frac{\Delta y}{\Delta x}$ when $x = -0.75$ and Δx has each of the following values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Use the Tangent checkbox to find the (approximate) slope of the tangent line to the graph of $f(x)$ at $x = -0.75$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

2. Now use the red slider to set $x = 0$.

   (a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0$ and Δx has the values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

3. Let $x = 0.75$.

   (a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0.75$ and Δx has the values:

   $$-0.5, -0.25, 0.25, 0.5.$$

   (b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0.75$.

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

4. Compare your answers to the previous problems.

   (a) Was your answer to part (c) the same for each problem?

   (b) For some values of x, $\frac{\Delta y}{\Delta x}$ was close to the slope of the tangent line when Δx was 0.5. For others it was not. Can you make any conjectures about when you need a very small value of Δx in order for $\frac{\Delta y}{\Delta x}$ to be close to the slope of the tangent line?

Solution

1. Move the red slider to $x = -0.75$; we'll investigate the slopes of secant lines passing through the point $(-0.75, f(-0.75))$.

   (a) Use the yellow slider to find the value of $\frac{\Delta y}{\Delta x}$ when $x = -0.75$ and Δx has each of the following values:

$$-0.5, -0.25, 0.25, 0.5.$$

Δx	$\dfrac{\Delta y}{\Delta x}$
-0.50	0.53
-0.25	0.16
0.25	-0.41
0.50	-0.59

   (b) Use the Tangent checkbox to find the (approximate) slope of the tangent line to the graph of $f(x)$ at $x = -0.75$.

$$m \approx -0.16.$$

   (c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.
   Any non-zero value of Δx with $-0.08 \leq \Delta x \leq 0.10$ is correct.

2. Now use the red slider to set $x = 0$.

   (a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0$ and Δx has the values:

$$-0.5, -0.25, 0.25, 0.5.$$

Δx	$\dfrac{\Delta y}{\Delta x}$
-0.50	-0.88
-0.25	-0.97
0.25	-0.97
0.50	-0.88

(b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0$.

$$m \approx -1.$$

(c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

Any non-zero value of Δx with $-0.46 \leq \Delta x \leq 0.46$ is correct.

3. Let $x = 0.75$.

(a) Find $\frac{\Delta y}{\Delta x}$ when $x = 0.75$ and Δx has the values:

$$-0.5, -0.25, 0.25, 0.5.$$

Δx	$\frac{\Delta y}{\Delta x}$
-0.50	-0.59
-0.25	-0.41
0.25	0.16
0.50	0.53

(b) Find the slope of the tangent line to the graph of $f(x)$ at $x = 0.75$.

$$m \approx -0.16.$$

(c) Find a value of Δx for which the value of $\frac{\Delta y}{\Delta x}$ is within 0.1 units of the slope of the tangent line.

Any non-zero value of Δx with $-0.10 \leq \Delta x \leq 0.08$ is correct.

Note: Because $f(x) = 0.5x^3 - x$ is an odd function, the answers to problem (3) are closely related to the answers to problem (1), and the answers to (2) are very symmetric. Later we will learn more about derivatives of odd and even functions.

4. Compare your answers to the previous problems.

(a) Was your answer to part (c) the same for each problem?

There was a much wider range of correct answers in (2) than in (1) and (3). It's likely that your answers to 1(c) and 3(c) were closer to 0 than your answer to 2(c) was.

Also notice that there was a wider range of positive correct answers to 1(c) and a wider range of negative correct answers to 3(c).

(b) For some values of x, $\frac{\Delta y}{\Delta x}$ was close to the slope of the tangent line when Δx was 0.25. For others it was not. Can you make any conjectures about when you need a very small value of Δx in order for $\frac{\Delta y}{\Delta x}$ to be close to the slope of the tangent line?

If we check the Tangent box then move the red slider without changing the (nonzero) value of Δx, we see that the angle between the tangent and secant lines is greatest where the graph of $f(x)$ is curved. If the graph of $y = f(x)$ is sharply curved, the value of Δx must be very close to 0 for the secant line to be close to the tangent line.

You may also have noticed that the difference between the slope of the secant and the slope of the tangent line was greater when the slope of the tangent line was large (and when x was large).

Note: Mathematicians frequently ask the question "how small does Δx have to be?" and have developed a suite of tools for answering this question in different contexts.