Natural log (inverse function of e^x)

Recall that:

$$M(a) = \lim_{\Delta x \to 0} \frac{a^{\Delta x} - 1}{\Delta x}.$$

is the value for which $\frac{d}{dx}a^x = M(a)a^x$, the value of the derivative of a^x when $x = 0$, and the slope of the graph of $y = a^x$ at $x = 0$. To understand $M(a)$ better, we study the natural log function $\ln(x)$, which is the inverse of the function e^x. This function is defined as follows:

$$\text{If } \ y = e^x, \ \text{ then } \ \ln(y) = x$$

or

$$\text{If } \ w = \ln(x), \ \text{ then } \ e^w = x$$

Before we go any further, let's review some properties of this function:

$$\ln(x_1 x_2) = \ln x_1 + \ln x_2$$

$$\ln 1 = 0$$

$$\ln e = 1$$

These can be derived from the definition of $\ln x$ as the inverse of the function e^x, the definition of e, and the rules of exponents we reviewed at the start of lecture.

We can also figure out what the graph of $\ln x$ must look like. We know roughly what the graph of e^x looks like, and the graph of $\ln x$ is just the reflection of that graph across the line $y = x$. Try sketching the graph of $\ln x$ yourself.

You should notice the following important facts about the graph of $\ln x$. Since e^x is always positive, the domain (set of possible inputs) of $\ln x$ includes only the positive numbers. The entire graph of $\ln x$ lies to the right of the y-axis. Since $e^0 = 1$, $\ln 1 = 0$ and the graph of $\ln x$ goes through the point $(1, 0)$. And finally, since the slope of the tangent line to $y = e^x$ is 1 where the graph crosses the y-axis, the slope of the graph of $y = \ln x$ must be $1/1 = 1$ where the graph crosses the x-axis.

We know that $\frac{d}{dx}e^x = e^x$. To find $\frac{d}{dx}\ln x$ we'll use implicit differentiation as we did in previous lectures.

We start with $w = \ln(x)$ and compute $\dfrac{dw}{dx} = \dfrac{d}{dx}\ln x$. We don't have a good way to do this directly, but since $w = \ln(x)$, we know $e^w = e^{\ln(x)} = x$. We now use implicit differentiation to take the derivative of both sides of this equation.

$$\begin{aligned} \frac{d}{dx}(e^w) &= \frac{d}{dx}(x) \\ \frac{d}{dw}(e^w)\frac{dw}{dx} &= 1 \end{aligned}$$

$$
\begin{aligned}
e^w \frac{dw}{dx} &= 1 \\
\frac{dw}{dx} &= \frac{1}{e^w} = \frac{1}{x}
\end{aligned}
$$

So

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

This is another formula worth memorizing.