Features of Graphs

The Graph Features mathlet allows you to choose the coefficients of a degree three polynomial and then illustrates where the graph of that polynomial is rising (increasing), falling (decreasing), concave and convex.

Find coefficient values a, b, c and d for a polynomial function:

$$f(x) = ax^3 + bx^2 + cx + d$$

whose graph is:

- convex for $x > 2$
- concave for $x < 2$
- falling when $x < 1$
- rising when $1 < x < 3$
- falling when $x > 3$.

Can you find two different polynomials that satisfy these requirements? Why or why not?

Bonus: Make up a problem similar to this one for a friend to solve.

Solution

We could use the mathlet and trial and error to find a function that matches this description. A more efficient method employs the concept of an "antiderivative", identifying a candidate for the second derivative and then working backward to find a function that satisfies the requirements of the problem.

From the concavity requirements, we see that the graph has a point of inflection at $x = 2$. It's natural to guess $f''(x) = x - 2$. Since the graph of $f(x)$ is concave for $x > 2$, the second derivative of f must be negative when $x > 2$, so we change our guess to:

$$f''(x) = -x + 2.$$

If $f''(x) = -x + 2$, what is f? For any constant k, the derivative of

$$f'(x) = -\frac{1}{2}x^2 + 2x + k$$

equals f''. We might consider $-\frac{1}{2}x^2 + 2x$, but f' must equal zero at the critical points 1 and 3. To achieve this you could compare the value of $f'(x)$ to the product $(x - 1)(x - 3)$ or plug in $x = 1$ (or $x = 3$) then solve for k:

$$\begin{aligned} f'(x) &= -\frac{1}{2}x^2 + 2x + k \\ f'(1) &= -\frac{1}{2} + 2 + k \end{aligned}$$

$$\begin{aligned} 0 &= \frac{3}{2} + k \\ k &= -\frac{3}{2} \end{aligned}$$

We conclude that we want:

$$f'(x) = -\frac{1}{2}x^2 + 2x - \frac{3}{2}.$$

Luckily, it's also true that $f'(x) = 0$ when $x = 3$:

$$\begin{aligned} f'(3) &= -\frac{1}{2} \cdot 9 + 2 \cdot 3 - \frac{3}{2} \\ &= -\frac{9}{2} + \frac{12}{2} - \frac{3}{2} \\ &= 0 \quad \checkmark \end{aligned}$$

The calculation of $f(x)$ proceeds similarly. First we find a general function whose derivative is $f'(x)$:

$$f(x) = -\frac{1}{6}x^3 + x^2 - \frac{3}{2}x + d.$$

Next we find the value of d. But wait! The value of d will have no effect on where the graph is rising, falling, concave or convex, so d can have any value. Let's use Professor Jerison's favorite number and choose $d = 0$.

We conclude that:

$$\begin{aligned} a &\approx -0.16 \\ b &= 1 \\ c &= -1.5 \\ d &= 0. \end{aligned}$$

We could find a different polynomial satisfying these requirements by selecting a different value of d, a different function $f''(x)$ which is zero at $x = 2$ and negative when $x > 2$, or by multiplying all of the coefficients by any non-zero number.

Use the mathlet to check this answer; because the value of a is approximate your curve may not match the description exactly.

Bonus: It's relatively easy to create a new problem of this type.

First, choose any linear expression $mx+b$ to be your second derivative $f''(x)$. The point of inflection of $f(x)$ will be at the point $x = -\frac{b}{m}$ and its concavity to the left and right of $x = -\frac{b}{m}$ will depend on the sign of m.

The function $f'(x) = \frac{m}{2}x^2 + bx + k$ then has derivative $f''(x)$. The critical points of $f(x)$ depend on your choice of k; if necessary you can use the quadratic formula to find them.

Finally,

$$f(x) = \frac{m}{6}x^3 + \frac{b}{2}x^2 + kx + d.$$

If you can't graph your function f using the mathlet, try multiplying or dividing all of its coefficients by the same constant.

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