REFLECTIONS IN A EUCLIDEAN SPACE

YOUR NAME HERE

18.099 - 18.06 CI.
Due on Monday, May 10 in class.

Write a paper proving the statements formulated below. Add your own examples, asides and discussions whenever needed.

Let V be a finite dimensional real linear space.

Definition 1. *A function* $\langle , \rangle : V \times V \to \mathbb{R}$ *is a* bilinear form *on* V *if for all* $x_1, x_2, x, y_1, y_2, y \in V$ *and all* $k \in \mathbb{R}$,

$$\langle x_1 + kx_2, y\rangle = \langle x_1, y\rangle + k\langle x_2, y\rangle, \text{ and}$$
$$\langle x, y_1 + ky_2\rangle = \langle x, y_1\rangle + k\langle x, y_2\rangle.$$

Definition 2. *A bilinear form* $\langle , \rangle$ *in* V *is* symmetric *if* $\langle x, y\rangle = \langle y, x\rangle$ *for all* $x, y \in V$. *A symmetric bilinear form is* nondegenerate *if* $\langle a, x\rangle = 0$ *for all* $x \in V$ *implies* $a = 0$. *It is* positive definite *if* $\langle x, x\rangle > 0$ *for any nonzero* $x \in V$. *An* inner product *on* V *is a symmetric positive definite bilinear form on* V.

Theorem 3. *Define a bilinear form on* $V = \mathbb{R}^n$ *by* $\langle e_i, e_j\rangle = \delta_{ij}$, *where* $\{e_i\}_{i=1}^n$ *is a basis in* V. *Then* $\langle , \rangle$ *is an inner product in* V.

Definition 4. *A* Euclidean space *is a finite dimensional real linear space with an inner product.*

Theorem 5. *Any* n*-dimensional Euclidean space* V *has a basis* $\{e_i\}_{i=1}^n$ *such that* $\langle e_i, e_j\rangle = \delta_{ij}$.

Hint: Use the Gram-Schmidt orthogonalization process.

Below $V = \mathbb{R}^n$ is a Euclidean space with the inner product $\langle , \rangle$.

Definition 6. *Two vectors* $x, y \in V$ *are* orthogonal *if* $\langle x, y\rangle = 0$. *Two subspaces* $U, W \in V$ *are orthogonal if* $\langle x, y\rangle = 0$ *for all* $x \in U$ *and* $y \in W$.

Check that if U and W are orthogonal subspaces in V, then $\dim(U) + \dim(W) = \dim(U + W)$.

Definition 7. *The* orthogonal complement *of the subspace* $U \subset V$ *is the subspace* $U^\perp = \{y \in V : \langle x, y\rangle = 0, \text{ for all } x \in U\}$.

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Definition 8. *A* hyperplane $H_x \subset V$ *is the orthogonal complement to the one-dimensional subspace in* V *spanned by* $x \in V$.

Theorem 9. *(Cauchy-Schwartz). For any* $x, y \in V$,

$$\langle x, y\rangle^2 \leq \langle x, x\rangle \cdot \langle y, y\rangle,$$

and equality holds if and only if the vectors x *and* y *are linearly dependent.*

We will be interested in the linear mappings that respect inner products.

Definition 10. *An* orthogonal operator *in* V *is a linear automorphism* $f : V \to V$ *such that* $\langle f(x), f(y)\rangle = \langle x, y\rangle$ *for all* $x, y \in V$.

Theorem 11. *If* f_1, f_2 *are orthogonal operators in* V, *then so are the inverses* f_1^{-1} *and* f_2^{-1} *and the composition* $f_1 \circ f_2$. *The identity mapping is orthogonal.*

Remark 12. *The above theorem says that orthogonal operators in a Euclidean space form a group, that is, a set closed with respect to compositions, containing an inverse to each element, and containing an identity operator.*

Example 13. *Describe the set of* 2×2 *matrices of all orthogonal operators in* $\mathbb{R}^2$, *and check that they form a group with respect to the matrix multiplication.*

Now we are ready to introduce the notion of a reflection in a Euclidean space. A reflection in V is a linear mapping $s : V \to V$ which sends some nonzero vector $\alpha \in V$ to its negative and fixes pointwise the hyperplane H_α orthogonal to α. To indicate this vector, we will write $s = s_\alpha$. The use of Greek letters for vectors is traditional in this context.

Definition 14. *A* reflection *in* V *with respect to a vector* $\alpha \in V$ *is defined by the formula:*

$$s_\alpha(x) = x - \frac{2\langle x, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha.$$

Theorem 15. *With the above definition, we have:*

(1) $s_\alpha(\alpha) = -\alpha$ *and* $s_\alpha(x) = x$ *for any* $x \in H_\alpha$;
(2) s_α *is an orthogonal operator;*
(3) $s_\alpha^2 = Id$.

Therefore, reflections generate a group: their compositions are orthogonal operators by Theorem 11, and an inverse of a reflection is equal to itself by Theorem 15. Below we consider some basic examples of subgroups of orthogonal operators obtained by repeated application of reflections.

Example 16. *Consider the group* S_n *of permutations of* n *numbers. It is generated by transpositions* t_{ij} *where* $i \neq j$ *are two numbers between* 1 *and* n, *and* t_{ij} *sends* i *to* j *and* j *to* i, *while preserving all other numbers.*

The compositions of all such transpositions form S_n. Define a set of linear mappings $T_{ij} : \mathbb{R}^n \to \mathbb{R}^n$ in an orthonormal basis $\{e_i\}_{i=1}^n$ by

$$T_{ij}e_i = e_j; \;\; T_{ij}e_j = e_i; \;\; T_{ij}e_k = e_k, k \neq i, j.$$

Then, since any element $\sigma \in S_n$ is a composition of transpositions, it defines a linear automorphism of $\mathbb{R}^n$ equal to the composition of the linear mappings defined above.

(1) *Check that T_{ij} acts as a reflection with respect to the vector $e_i - e_j \in \mathbb{R}^n$.*
(2) *Check that any element σ of S_n fixes pointwise the line in $\mathbb{R}^n$ spanned by $e_1 + e_2 + \ldots e_n$.*
(3) *Let $n = 3$. Describe the action of each element (how many are there?) of S_3 in $\mathbb{R}^3$ and in the plane U orthogonal to $e_1 + e_2 + e_3$. Example 13 lists all matrices of orthogonal operators in $\mathbb{R}^2$. Identify among them the matrices corresponding to the elements of S_3 acting in U. Check that the product of two reflections is a rotation.*

Example 17. *The action of S_n in $\mathbb{R}^n$ described above can be composed with the reflections $\{P_i\}_{i=1}^n$, sending e_i to its negative and fixing all other elements of the basis $e_k, k \neq i$.*

(1) *Check that the obtained set of orthogonal operators has no nonzero fixed points (elements $x \in \mathbb{R}^n$ such that $f(x) = x$ for all f in the set).*
(2) *How many distinct orthogonal operators can be constructed in this way for $n = 2$ and $n = 3$?*
(3) *In case $n = 2$, identify the matrices of the obtained orthogonal operators among those listed in Example 13.*

Remark 18. *The two examples above correspond to the series A_{n-1} and B_n in the classification of finite reflection groups.*