CARTAN MATRICES, DYNKIN DIAGRAMS AND CLASSIFICATION

YOUR NAME HERE

18.099 - 18.06 CI.
Due on Monday, May 10 in class.

Write a paper proving the statements and working through the examples formulated below. Add your own examples, asides and discussions whenever necessary.

Let V be a Euclidean space, that is a finite dimensional real linear space with a symmetric positive definite inner product $\langle , \rangle$.

Recall that for a root system Δ in V, there exists a simple root system $\Pi \subset \Delta$ (not unique), which is a basis in V, and a positive root system Δ^+ such that any positive root is a linear combination of simple roots with nonnegative integer coefficients.

Below we will assume that the root system Δ is reduced (that is, for any $\alpha \in \Delta$, $2\alpha \notin \Delta$).

For a given pair (Π, Δ) with a fixed enumeration of simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where $l = \dim(V)$, we defined in [5] the Cartan matrix A by setting

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

An abstract Cartan matrix is an $l \times l$ matrix with the following properties:

(1) all entries A_{ij} are integers;
(2) $A_{ii} = 2$ for all i;
(3) $A_{ij} \leq 0$ for all $i \neq j$;
(4) $A_{ij} = 0$ if and only if $A_{ji} = 0$;
(5) there exists a diagonal matrix D with positive entries such that DAD^{-1} is symmetric positive definite.

A Cartan matrix is irreducible if it is not isomorphic (conjugate by a product of permutation matrices) to a block diagonal matrix with more than one block.

We know that a Cartan matrix determines the simple root system uniquely up to isomorphism ([5]). Recall that two root systems Δ and Δ' in V are isomorphic if there exists a linear automorphism of V that maps Δ onto Δ' preserving the numbers $n(\alpha, \beta)$ for all $\alpha, \beta \in \Delta$. The goal of this paper is to

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show that any abstract Cartan matrix corresponds to a reduced root system and determines it up to isomorphism. This provides a tool for classification of abstract root systems. The proof requires several steps.

Theorem 1. *Let* $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_l\} \subset \Delta$ *be a set of simple roots in a reduced root system* Δ*. For any root* $\alpha \in \Delta$*, there exists an element* $\alpha_j \in \Pi$ *such that* $\alpha = w(\alpha_j)$*, where* w *is a composition of reflections with respect to simple roots in* Π.

Hint: First assume that $\alpha = \sum_{i=1}^{l} n_i \alpha_i$ is a positive root and proceed by induction in the number $\sum_{i=1}^{l} n_i$ (the level of a root). Use Theorem 5 in [4] to find an element $\alpha_k \in \Pi$ such that the reflection with respect to α_k maps α to a positive root with a smaller level. This is the induction step. Then extend the result for the negative roots, using that the reflection with respect to $\alpha_i \in \Pi$ maps α_i to $-\alpha_i$.

Theorem 2. *The set* $\{\alpha_1, \alpha_2, \dots, \alpha_l\}$ *of simple roots determines the set of all roots in a reduced root system.*

Hint: use Theorem 1.

The last Theorem together with Theorem 11 in [5] shows that an abstract Cartan matrix corresponds to at most one reduced root system, up to isomorphism. Using the properties of abstract Cartan matrices, it is possible to further show that *every* abstract Cartan matrix determines a reduced abstract root system. Consequently, it is possible to classify all reduced root systems based on the properties of Cartan matrices. Here is an example in dimension 3.

Example 3. *Find all* 3×3 *abstract Cartan matrices up to isomorphism and construct the corresponding root systems. Using examples in* [2, 3, 4, 5]*, identify the type of a root system whenever possible.*

Hint: Start with the block diagonal Cartan matrices with more than one block. The properties (1)-(4) of an abstract Cartan matrix together with the conditions on the numbers $n(\alpha, \beta) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ for simple roots, discussed at the end of [2], provide sufficient information to classify all such Cartan matrices. For the irreducible 3×3 Cartan matrices, use that all upper left determinants of a positive definite matrix are positive.

A similar argument in higher dimensions eventually leads to a complete classification of the abstract (reduced) root systems in Euclidean spaces.

Continue this paper with a review of literature (without proofs) on abstract root systems, in particular define the Dynkin diagram of a root system and formulate the complete classification of abstract irreducible root systems. Suggested sources: [6], *§5.12-5.15,* [7], *§5.8-5.9,* [8], *§4.5. Any one of these sourses will be sufficient.*

References

[2] Your classmate, *Abstract root systems*, preprint, MIT, 2004.

[3] Your classmate, *Simple and positive roots*, preprint, MIT, 2004.
[4] Your classmate, *Properties of simple roots*, preprint, MIT, 2004.
[5] Your classmate, *Cartan matrix of a root system*, preprint, MIT, 2004.
[6] J.-P. Serre, *Complex semisimple Lie algebras*, New York, Springer-Verlag, 1987
[7] W.A. De Graaf, *Lie algebras: theory and algorithms*, Amsterdam, New York, Elsevier, 2000
[8] N. Jacobson, *Lie algebras*, New York, Dover, 1979