PROPERTIES OF SIMPLE ROOTS

YOUR NAME HERE

18.099 - 18.06 CI.
Due on Monday, May 10 in class.

Write a paper proving the statements and working through the examples formulated below. Add your own examples, asides and discussions whenever needed.

Let V be a Euclidean space, that is a finite dimensional real linear space with a symmetric positive definite inner product $\langle , \rangle$.

Recall that for a root system Δ in V, a subset $\Pi \subset \Delta$ is a set of simple roots (a simple root system) if

(1) Π is a basis in V;
(2) Each root $\beta \in \Delta$ can be written as a linear combination of elements of Π with integer coefficients of the same sign, i.e.

$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$$

with all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$.

The root β is positive if the coefficients are nonnegative, and negative otherwise. The set of all positive roots (positive root system) associated to Π is denoted Δ^+.

Below we will assume that the root system Δ is reduced, that is, for any $\alpha \in \Delta, 2\alpha \notin \Delta$.

Theorem 1. *In a given Δ, a set of simple roots $\Pi \subset \Delta$ and the associated set of positive roots $\Delta^+ \subset \Delta$ determine each other uniquely.*

Hint: easy. Use the explicit construction of $\Pi \subset \Delta^+$ given in [3].

The question of existence of sets of simple roots for any abstract root system Δ is settled in [3]. Theorem 1 shows that once Π is chosen Δ^+ is unique. In this paper we want to address the question of the possible choices for $\Pi \subset \Delta$. We start with a couple of examples.

Example 2. *The root system of the type A_2 consists of the six vectors $\{e_i - e_j\}_{i \neq j}$ in the plane orthogonal to the line $e_1 + e_2 + e_3$ where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$. Present the vectors of this root system in a standard orthonormal basis of the plane. Find possible simple root systems*

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$\Pi \subset \Delta$ and the associated sets of positive roots Δ^+, $\Pi \subset \Delta^+ \subset \Delta$. Check that any two simple root systems $\Pi \subset \Delta$ can be mapped to each other by an orthogonal transformation (see [1] for definition) of V, and that the same transformation maps the associated sets of positive roots.

Example 3. *Consider the root system of the type B_2 in $V = \mathbb{R}^2$: it consists of eight vectors $\{\pm e_1 \pm e_2, \pm e_1, \pm e_2\}$. Find possible simple root systems $\Pi \subset \Delta$ and check that they can be obtained from any chosen one by an orthogonal transformation of $\mathbb{R}^2$. Check that the same transformation maps the associated sets of positive roots to each other.*

We start working towards a result generalizing our observations. Recall the definition of a reflection associated to an element $\alpha \in V$ (cf. [1]):

$$s_\alpha(x) = x - \frac{2\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

It is an orthogonal transformation of V.

Theorem 4. *Let $\Pi \subset \Delta$ be a set of simple roots, associated to the set of positive roots Δ^+. For any $\alpha \in \Delta$, the set obtained by reflection $s_\alpha(\Pi)$ is a simple root system with the associated positive root system $s_\alpha(\Delta^+)$.*

To understand better the passage from Δ^+ to $s_\alpha(\Delta^+)$, we consider the special case when α is a simple root. Then Δ^+ and $s_\alpha(\Delta^+)$ differ by only one root:

Theorem 5. *Let $\Pi \subset \Delta$ be a simple root system, contained in a positive root set Δ^+. If $\alpha \in \Pi$, then the reflection s_α maps the set $\Delta^+ \setminus \{\alpha\}$ to itself.*

Corollary 6. *Any two positive root systems in Δ can be obtained from each other by a composition of reflections with respect to the roots in Δ.*

Hint: Let Δ_1^+ and Δ_2^+ be two positive root systems. Recall that the negative roots Δ_i^- are the negatives of the elements in Δ_i^+, $i = 1, 2$ (see [3]). Use induction on the number of elements in the intersection $\Delta_1^+ \cap \Delta_2^-$. Theorem 5 provides a way to decrease this number by one.

The statements above show that although a set of simple roots is not unique for a given Δ, they are related to each other by a simple orthogonal transformation of the space V. In particular, the angles and relative lengths of simple roots in any two simple root systems in Δ are the same. The next theorem proves another useful property of simple roots.

Theorem 7. *Let $\Pi \subset \Delta^+$ be a simple and a positive root systems in Δ. Any positive root $\beta \in \Delta^+$ can be written as a sum*

$$\beta = \alpha_1 + \alpha_2 + \ldots + \alpha_k,$$

where $\alpha_i \in \Pi$ for all $i = 1, \ldots, k$ (repetitions are allowed). Moreover, it can be done so that each partial sum

$$\alpha_1 + \ldots + \alpha_m, \quad 1 \leq m \leq k$$

is also a root.

Hint: Choose $t \in V$ such that $\langle t, \alpha \rangle = 1$ for all $\alpha \in \Pi$. Prove that such t exists and that the number $r = \langle t, \beta \rangle$ is a positive integer for any $\beta \in \Delta^+$. Using Lemma 7 in [3], show that $\langle \alpha, \beta \rangle > 0$ for some $\alpha \in \Pi$. Then proceed by induction on r. Theorem 10(1) in [2] allows us to reduce r by one.

Example 8. *Let Δ be the root system in $V = \mathbb{R}^2$ such that the angle between the simple roots is $\frac{5\pi}{6}$. This condition determines Δ completely (this is the root system of the type G_2). Construct and sketch the simple roots, positive roots, and the whole root system Δ. Apply Theorem 7 in this case to present each positive root as a sum of simple roots.*

Recall that two root systems Δ and Δ' are *isomorphic* if there exists an linear automorphism of V that maps Δ onto Δ' and preserves the integers $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} i$. A root system is irreducible if it cannot be decomposed as a disjoint union of two root systems $\Delta = \Delta' \cup \Delta''$ of smaller dimension, so that each element of Δ' is orthogonal to each element of Δ''.

Example 9. *Up to isomorphism, there are just three reduced irreducible root systems in $V = \mathbb{R}^3$, of the types A_3, B_3 and C_3 (see Example 5 in* [2]*, Examples 10 and 11 in* [3] *for definitions). Find the other possible reduced root systems in $V = \mathbb{R}^3$ (they can be represented as a union of two or more root systems in smaller dimensions).*

Hint: Note that the only reduced root system in $\mathbb{R}$ is of the type A_1. A classification of root systems in $\mathbb{R}^2$ can be carried out as indicated at the end of [2].

References

[1] Your classmate, *Reflections in a Euclidean space*, preprint, MIT, 2004.
[2] Your classmate, *Abstract root systems*, preprint, MIT, 2004.
[3] Your classmate, *Simple and positive roots*, preprint, MIT, 2004.