SIMPLE AND POSITIVE ROOTS

YOUR NAME HERE

18.099 - 18.06 CI.

Due on Monday, May 10 in class.

Write a paper proving the statements and working through the examples formulated below. Add your own examples, asides and discussions whenever needed.

Let V be a Euclidean space, that is a finite dimensional real linear space with a symmetric positive definite inner product $\langle , \rangle$.

Recall that a root system in V is a finite set Δ of nonzero elements of V such that

(1) Δ spans V;
(2) for all $\alpha \in \Delta$, the reflections

$$s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$$

map the set Δ to itself;
(3) the number $\frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$ is an integer for any $\alpha, \beta \in \Delta$.

A root is an element of Δ.

Here are two examples of root systems in $\mathbb{R}^2$:

Example 1. *The root system of the type $A_1 \oplus A_1$ consists of the four vectors $\{\pm e_1, \pm e_2\}$ where $\{e_1, e_2\}$ is an orthonormal basis in $\mathbb{R}^2$.*

Example 2. *The root system of the type A_2 consists of the six vectors $\{e_i - e_j\}_{i \neq j}$ in the plane orthogonal to the line $e_1 + e_2 + e_3$ where $\{e_1, e_2, e_3\}$ is an orthonormal basis in $\mathbb{R}^3$. Rewrite the vectors of this root system in a standard orthonormal basis of the plane and sketch it.*

Since for any $\alpha \in \Delta$, $-\alpha$ is also in Δ, (see [1], Thm.8(1)), the number of elements in Δ is always greater than the dimension of V. The example of type A_2 above shows that even a subset of mutually noncollinear vectors in Δ might be too big to be linearly independent. In the present paper we would like to define a subset of Δ small enough to be a basis for V, yet large enough to contain the essential information about the geometric properties of Δ. Here is a formal definition.

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Definition 3. *A subset* Π *of* Δ *is a* set of simple roots *(a* simple root system*) in* Δ *if*

(1) Π *is a basis for* V*;*
(2) *Each root* $\beta \in \Delta$ *can be written as a linear combination of the elements of* Π *with integer coefficients of the same sign, that is,*

$$\beta = \sum_{\alpha \in \Pi} m_\alpha \alpha$$

with all $m_\alpha \geq 0$ *or all* $m_\alpha \leq 0$.

The root β *is* positive *if the coefficients are nonnegative, and* negative *otherwise. The set of all positive roots (the* positive root system*) associated to* Π *will be denoted* Δ^+.

Below we construct a set Π_t associated to an element $t \in V$ and a root system Δ, and show that it satisfies the definition of a simple root system in Δ.

Let Δ be a root system in V, and let $t \in V$ be a vector such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$ (Check that such an element always exists). Set

$$\Delta_t^+ = \{\alpha \in \Delta : \langle t, \alpha \rangle > 0\}.$$

Let $\Delta_t^- = \{-\alpha, \alpha \in \Delta_t^+\}$. Check that $\Delta = \Delta_t^+ \cup \Delta_t^-$.

Definition 4. *An element* $\alpha \in \Delta_t^+$ *is* decomposable *if there exist* $\beta, \gamma \in \Delta_t^+$ *such that* $\alpha = \beta + \gamma$. *Otherwise* $\alpha \in \Delta_t^+$ *is* indecomposable.

Let $\Pi_t \subset \Delta_t^+$ be the set of all indecomposable elements in Δ_t^+.

The next three Lemmas prove the properties of Δ_t^+ and Π_t.

Lemma 5. *Any element in* Δ_t^+ *can be written as a linear combination of elements in* Π_t *with nonnegative integer coefficients.*

Hint: By contradiction. Suppose γ is an element of Δ_t^+ for which the Lemma is false and $\langle t, \gamma \rangle > 0$ is minimal, and use that γ is decomposable to get a contradiction.

Lemma 6. *If* $\alpha, \beta \in \Pi_t$, *then* $\langle \alpha, \beta \rangle \leq 0$.

Hint: Use Thm. 10(1) in [1] : if $\langle \alpha, \beta \rangle > 0$, then $\alpha - \beta$ is a root or 0.

Add discussion: what does this result mean for the relative position of two simple roots?

Lemma 7. *Let* A *be a subset of* V *such that*

(1) $\langle t, \alpha \rangle > 0$ *for all* $\alpha \in A$*;*
(2) $\langle \alpha, \beta \rangle \leq 0$ *for all* $\alpha, \beta \in A$.

Then the elements of A *are linearly independent.*

Hint: Assume the elements of A are linearly dependent and split the nontrivial linear combination into two sums, with positive and negative coefficients. Let $\lambda = \sum m_\beta \beta = \sum n_\gamma \gamma$ with $\beta, \gamma \in A$ and all $m_\beta, n_\gamma > 0$. Show that $\langle \lambda, \lambda \rangle = 0$.

Now we are ready to prove the existence of a simple root set in any abstract root system.

Theorem 8. *For any $t \in V$ such that $\langle t, \alpha \rangle \neq 0$ for all $\alpha \in \Delta$, the set Π_t constructed above is a set of simple roots, and Δ_t^+ the associated set of positive roots.*

Hint: Use lemmas 5, 6, 7.

The converse statement is also true (and much easier to prove):

Theorem 9. *Let Π be a set of simple roots in Δ, and suppose that $t \in V$ is such that $\langle t, \alpha \rangle > 0$ for all $\alpha \in \Pi$. Then $\Pi = \Pi_t$, and the associated set of positive roots $\Delta^+ = \Delta_t^+$.*

Example 10. *Let V be the n-dimensional subspace of $\mathbb{R}^{n+1}$ $(n \geq 1)$ orthogonal to the line $e_1 + e_2 + \ldots + e_{n+1}$, where $\{e_i\}_{i=1}^{n+1}$ is an orthonormal basis in $\mathbb{R}^{n+1}$. The root system Δ of the type A_n in V consists of all vectors $\{e_i - e_j\}_{i \neq j}$. Check that $\Pi = \{e_1 - e_2, e_2 - e_3, \ldots e_n - e_{n+1}\}$ is a set of simple roots, and $\Delta^+ = \{e_i - e_j\}_{i<j}$ - the associated set of positive roots in Δ.*

Example 11. *The root system Δ of the type C_n in $V = \mathbb{R}^n$ $(n \geq 2)$ consists of all vectors $\{\pm e_i \pm e_j\}_{i \neq j} \cup \{\pm 2e_i\}$, where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis in $\mathbb{R}^n$. Check that $\Pi = \{e_1 - e_2, e_2 - e_3, \ldots e_{n-1} - e_n, 2e_n\}$ is a set of simple roots, and $\Delta^+ = \{e_i \pm e_j\}_{i<j} \cup \{2e_i\}$ - the associated set of positive roots in Δ.*

Example 12. *Let $V = \mathbb{R}^2$ and recall from [1], that for any two roots α, β,*

$$n(\alpha, \beta) \cdot n(\beta, \alpha) = 4\cos^2(\phi),$$

where $n(\alpha, \beta) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}$, and ϕ is the angle between α and β. Using Lemma 6, find all possible angles between the simple roots in $\mathbb{R}^2$, and their relative lengths. Sketch the obtained pairs of vectors. Identify those that correspond to the root systems $A_1 \oplus A_1$, A_2 and C_2 discussed in Examples 1, 2 and 11 for $n = 2$. In these three cases, describe and sketch the set of all elements $t \in V$ such that $\Pi_t = \Pi$ for a given Π. This set is the dominant Weyl chamber for (Δ, Π).

References

[1] Your classmate, *Abstract root systems*, preprint, MIT, 2004.