CARTAN MATRIX OF A ROOT SYSTEM

YOUR NAME HERE

18.099 - 18.06 CI.
Due on Monday, May 10 in class.

Write a paper proving the statements and working through the examples formulated below. Add your own examples, asides and discussions whenever needed.

Let V be a Euclidean space, that is a finite dimensional real linear space with a symmetric positive definite inner product $\langle , \rangle$.

Recall that for a root system Δ in V, there exists a simple root system $\Pi \subset \Delta$ (not unique), such that

(1) Π is a basis in V;
(2) Each root $\beta \in \Delta$ can be written as a linear combination of elements of Π with integer coefficients of the same sign.

The root β is positive if the coefficients are nonnegative, and negative otherwise. The set of all positive roots (the positive root system) associated to Π is denoted Δ^+.

Below we will assume that the root system Δ is reduced (that is, for any $\alpha \in \Delta$, $2\alpha \notin \Delta$).

We shall associate a "Cartan matrix" to the system $\Pi \subset \Delta$ and derive some properties of this matrix. An abstract Cartan matrix will be any square matrix with this list of properties. It turns out that an abstract Cartan matrix essentially determines the root system. In this paper we will work toward making this statement precise and proving it. The problem of classification of root systems is reduced then to the classification of the Cartan matrices.

Definition 1. *Let $\Pi \subset \Delta$ be a chosen set of simple roots in the root system Δ, and suppose that the simple roots are enumerated by $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$, where $l = \dim(V)$. The* Cartan matrix *of (Π, Δ) is the $l \times l$ matrix A with*

$$A_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

The Cartan matrix clearly depends on the enumeration of Π, but this dependence can be easily sorted out. Recall that a permutaion matrix P^{ij} is a square matrix with $1 \le i \ne j \le l$, which is obtained from the identity

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matrix by exchanging rows i and j. Note that a permutation matrix is always nonsingular. We need the following easy fact from linear algebra:

Theorem 2. *For any $l \times l$ matrix B, $B' = P^{ij}B(P^{ij})^{-1}$ is the matrix B with i-th and j-th rows and columns exchanged.*

Corollary 3. *Let A be the Cartan matrix associated to (Π, Δ) with a fixed enumeration of the elements of Π, and let A' be the Cartan matrix of the same root system with α_i and α_j exchanged in the enumeration of the simple roots. Then $A' = P^{ij}A(P^{ij})^{-1}$.*

Definition 4. *Two Cartan matrices are* isomorphic *if they are conjugate by a product of permutation matrices.*

To understand the properties of A, we start with some examples.

Example 5. *Let (Π, Δ) be the root system of type A_n (see Example 10, [2]) with the simple roots enumerated as listed there. Find the Cartan matrix of this system.*

Example 6. *Let Δ be a root system of the type B_4 with the root vectors $\{\pm e_i \pm e_j\}_{i \neq j} \cup \{\pm e_i\}$, where $\{e_i\}_{i=1}^4$ is an orthonormal basis in $\mathbb{R}^n$. Check that $\Pi = \{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4\}$ is a set of simple roots, and $\Delta^+ = \{e_i \pm e_j\}_{i<j} \cup \{e_i\}$ - the associated set of positive roots in Δ. Find the Cartan matrix of this root system.*

We summarize the observed properties in the following statement. Recall that a square matrix B is symmetric if $B_{ij} = B_{ji}$ for all i, j. A symmetric matrix C is positive definite if $\langle C \cdot x, x\rangle > 0$ for any nonzero $x \in V$.

Theorem 7. *The Cartan matrix A of a root system (Π, Δ) has the following properties:*

1. *all entries A_{ij} are integers;*
2. *$A_{ii} = 2$ for all i;*
3. *$A_{ij} \leq 0$ for all $i \neq j$;*
4. *$A_{ij} = 0$ if and only if $A_{ji} = 0$;*
5. *there exists a diagonal matrix D with positive entries such that DAD^{-1} is symmetric positive definite.*

Hint: The only nontrivial statement is the last one. Try $D = diag(|\alpha_1|, \dots |\alpha_l|)$, where $|\alpha_i| = \langle \alpha_i, \alpha_i\rangle^{1/2}$ is the length of a simple root.

Example 8. *Recall that two root systems in the same vector space are* isomorphic *if they can be mapped to each other by a linear transformation preserving angles between the roots and relative lengths within each irreducible component. Classify the reduced root systems in $V = \mathbb{R}^2$ (follow the method indicated at the end of [1]). Choose a simple root system in each case and find the Cartan matrix. Whenever necessary, conjugate by a diagonal matrix D and check that the result is symmetric positive definite. Note that DAD^{-1} may have non-integer entries. Is the matrix D determined uniquely by the condition (5) in Theorem 7?*

Recall from [1] that a root system is *irreducible* if it doesn't admit a decomposition into two root systems $\Delta = \Delta' \cup \Delta''$, where each element of Δ' is orthogonal to each element of Δ''. If $\Delta = \Delta' \cup \Delta''$ and Δ' is irreducible, then it is an irreducible component of Δ. A root system is said to be reducible if it is not irreducible. Irreducibility of a root system can be detected on the level of Cartan matrices.

Theorem 9. *A reduced root system is reducible if and only if for some choice of a simple root system and some enumeration of indices, the Cartan matrix is block diagonal with more than one block.*

Hint: The nontrivial part is to show that a root system corresponding to a block diagonal Cartan matrix A is reducible. Suppose that A has two blocks corresponding to the simple roots $\{\alpha_1, \dots, \alpha_r\}$ and $\{\alpha_{r+1}, \dots, \alpha_l\}$ respectively. For any $\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Delta$ you need to show that α belongs to one of the irreducible components of Δ spanned by the simple roots from one of the blocks. Assume that α is positive and proceed by induction in the number $\sum_{i=1}^{l} n_i$. You might need the property of root systems given in Theorem 10(2) in [1].

A Cartan matrix such that any isomorphic Cartan matrix has a single block is said to be irreducible. Now we are ready to answer the question of uniqueness of D in general.

Theorem 10. *The matrix D is determined uniquely up to a scalar multiple on each block of A.*

Hint: Straightforward linear algebra. Suppose there are two diagonal matrices D and D' which symmetrize a Cartan matrix A, and compare the entries.

Note that by Theorem 7 (5), in the case that A is irreducible, the matrix D gives the relative lengths of the simple roots. Then the matrix elements of A determine the relative angles between any two simple roots. We deduce the following

Corollary 11. *The Cartan matrix for a set of simple roots determines that set of simple roots uniquely up to root system isomorphism; that is, up to a scalar multiple of an orthogonal transformation on each irreducible component.*

Example 12. *Let $\Delta = \Delta' \cup \Delta''$ be the union of the root systems Δ' of type A_1 and Δ'' of type B_2. (See Example 5 above for A_n and Example 3 in* [3] *for B_2). Find the Cartan matrix of this root system, and use a diagonal matrix D to symmetrize it.*

References

[1] Your classmate, *Abstract root systems*, preprint, MIT, 2004.
[2] Your classmate, *Simple and positive roots*, preprint, MIT, 2004.
[3] Your classmate, *Properties of simple roots*, preprint, MIT, 2004.