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CASEY OK, so the name of the game today is compactness. And so last time I recalled that the definition of a subset of a metric space X is compact if every sequence x_n in K , so every sequence of elements in K has a subsequence converging in K , has a subsequence which converges to an element in K .

And so for example by the Heine-Borel theorem, closed and bounded subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are compact. This is from 18100. And general closed and bounded subsets of a Hilbert space are not necessarily compact. Last time we looked at the closed unit ball in the Hilbert space, or let's say in little ℓ_2 , this is not compact because you can take orthonormal basis vectors which are the sequences that have zero except for one in the n -th spot.

These are all of unit length. And the sequence consisting of these basis vectors, which are sequences, so the sequence of sequences does not converge in little ℓ_2 or have a subsequence which converges in little ℓ_2 .

So we were looking for an additional condition that would ensure compactness. And as a warm up to this, last time we proved the following that at-- so we're always working in a Hilbert space that may be separable or not depending on if I say it is or not-- that if we take a sequence, a convergent sequence in H , then the conclusion is, 1, that the set K consisting of all the elements of this sequence and the limit is compact.

And the set K has what are called equi-small tails with respect to any orthonormal subset of H . If e is a countable orthonormal subset of H , then the following holds, this condition we called K having equi-small tails, then for every ϵ positive, there exists a natural number N such that for all-- let's call it-- use a different letter, v in K , so either an element of the sequence or v , we have that the sum over K bigger than N squared is less than ϵ squared.

So what this says in this condition we called K has equi-small tails with respect to this orthonormal subset. So of course for any fixed v , because this entire sum-- forget just K over N -- is bounded by the norm of this v squared by Bessel's inequality. We can always choose an N for any individual v .

Now equi-small tails means that I can choose this independent of the element in K . So if you like uniformly small tails is another way you could think about it. But the terminology equi has been used instead. But another way is to think we can uniformly make the tails with respect to an orthonormal sequence, or orthonormal subset of H small.

Now, what we're going to prove is that, in fact, this condition here of having equi-small tails with respect to an orthonormal basis on top of the set being closed and bounded, suffice to prove that the subset is compact. And that's the following theorem. So let H be a separable Hilbert space.

And let e_k be an orthonormal basis of H . So I said suffice, but not just suffice. It's also necessary, as we'll show. Then K subset of H is compact if and only if K is closed, bounded, and has equi-small tails with respect to this orthonormal basis, so in particular for any orthonormal basis.

So this is a complete description of compact subsets of a Hilbert space, it's those subsets which are closed, bounded, and have equi-small details with respect to one orthonormal basis. And what this theorem also says is that if it has equi-small details with respect to one orthonormal basis and is closed and bounded, then the same holds with respect to any orthonormal basis.

OK, so let's prove-- so it's a two way street. Well, it doesn't make sense to say one street is shorter, one side of the street is shorter-- maybe wider, easier to use. But let's prove that if K is compact, then it implies all of this. So suppose K is compact and K is closed and bounded by general metric space theory. So look this up in the 18100's material, or intro to analysis material.

So let's-- this also gives us an opportunity to understand this equi-small tail condition as well by having to negate it. So we're also going to prove that K has equi-small details by contradiction. So suppose K does not have equi-small tails with respect to this basis, or this orthonormal basis.

Then what's that mean? So this is a for all there exists in a statement. So then there exists some bad epsilon so that for all N , a natural number, there exist a u_N in K such that-- so these N 's are increasing such that u_N -- so we have the negation of the condition that satisfied K bigger than N of u_N ϵ_K squared is bigger than or equal to this epsilon 0 squared.

So note that this is a sequence in this compact set K . And so by the defining property of a compact set, it must have a convergent subsequence, which implies there exists subsequence which you usually denote use of N , capital N sub m , some other letter. But I'll just call it, to make it a little connection back to what we wrote a minute ago, v sub n .

So this is a subsequence of u sub capital N and v in K such that v sub n converges to v . Now then for all N , natural number, we have that sum over K bigger than n of v_N ϵ_K squared is bigger than epsilon 0 squared because this is a subsequence of the original sequence which satisfied this condition.

Now this would actually be capital N sub n . But if I throw in some more, capital N sub n would be an increasing sequence of numbers. And it's always bigger than n . So I can replace that capital N sub n with this. All right, so this is fine.

Then that says the subset consisting of the elements v sub n in the natural number union v does not have equi-small tails, which is a contradiction to the theorem we proved last time. This contradicts the previous theorem that I stated and we proved last time.

So any convergent sequence along with its limit is a compact set and has equi-small tails. So that's a contradiction. And what was the assumption that brought us to this road forking itself is the assumption that K is not compact. So therefore K -- I mean, not K 's compact. K has equi-small tails with respect to this orthonormal basis.

So that's one direction that if K is compact then it has to have equi-small tails with respect to the orthonormal basis. We didn't use anything about it being a basis, so this also says that if you have a compact set, then with respect to any orthonormal subset, it has equi-small tails with respect to that orthonormal subset.

So now let's use this condition that K is closed and bounded and has equi-small tails to prove that K is compact. So suppose K is closed, bounded, and has equi-small tails with respect to this orthonormal basis.

Now let u_N be a sequence in K . So we want to show that it has a subsequence which converges to an element in K . Now, K is closed, which means that if I take any sequence which converges in K , or which any sequence of elements in K that converges, the limit has to be in K . That's the definition of a set being closed.

So I just need to show that this sequence, in fact, has a convergence subsequence. That's all I have to show. So since K is closed, we just need to show u_N as a convergent subsequence. So the idea is to use what we know about bounded subsets of complex numbers. So we know about bounded subsets of complex numbers.

If I have a bounded sequence-- I shouldn't say subsets, but if I have a bounded sequence of complex numbers, it has a convergent subsequence. This is a Bolzano-Weierstrass. Usually it's stated in terms of subsets of $\mathbb{R}$. But sequences of complex numbers converge if and only if the real and imaginary part converge, and are bounded if and only if the real and imaginary part, which are sequences of real numbers, are bounded.

So Bolzano-Weierstrass immediately implies a similar statement for complex numbers, that every bounded sequence of complex numbers has a convergent subsequence. so. How would we use that? What we're going to do is we're going to expand u_N 's. And this is where we'll use that this as an orthonormal basis in terms of this orthonormal basis.

So for each K , we'll have a sequence of complex numbers given by the entry, the K -th entry, of course the K -th entry of this sequence. And that will be a bounded sequence of complex numbers because we're assuming K is bounded, and therefore this sequence is bounded in K .

So what we're going to end up having is entry by entry, a bounded sequence of complex numbers where these complex numbers represent the K -th entry of the sequence of vectors. And we're going to then take a subsequence that converges for the first entry. We're going to take a subsequence of that subsequence to get a subsequence which converges for the first and second entry, and then so on and so on, a diagonalization argument. And that subsequence we'll show converges.

Now, it's not that simple because if you can do something 10 times it doesn't mean you can get some uniform control. We just have control for any finite number of entries. What saves us in the end is this equi-small tail condition, which allows us to basically throw away infinitely many entries.

That's what this says is that you can always choose an N , so that the tail end of the entries don't matter. And so that, with having control over the first large block of entries, will be enough to conclude. So that was the roadmap. If you didn't follow the roadmap, fine. Hopefully you follow the proof.

But it's a diagonal argument. And those are relatively easy to understand, somewhat difficult to write down, so I will do my best. So since K is bounded, that implies there exists a c non-negative such that for all n , norm of $u_{\text{sub } n}$ is less than or equal to C . The C is independent of n . It just depends on the set K .

And therefore that sets for all K and n -- I should say for all K , for all n , if I look at the K -th-- if you let me call this the Fourier coefficient in this basis in this orthonormal basis of e_k , if I look at the K -th Fourier coefficient, this is bounded by Cauchy-Schwartz, $u_n e_k$.

And these are orthonormal so this has unit length. This is bounded by c . So for each fixed K , this is a sequence in N of complex numbers which is bounded. The sequence consisting of the K -th Fourier coefficient of the elements $u_{\text{sub } n}$, so n is the thing that's changing, is the bounded sequence of complex numbers.

So now this is where we're going to start choosing subsequences of subsequences so that we get convergence along the entries. So since u_n is bounded, this implies that there exists by Bolzano-Weierstrass, a subsequence, which I'll denote u_{n_1} of $k \rightarrow \infty$. Let me not use k . Let's use j . So u_{n_1} is ϵ_1 , so $1, 1, \dots$ in n , which converges in c .

OK, so I now have a subsequence of the original sequence of the u_n 's so that the first entry converges here. Or this sequence converges. Now I'm going to take-- OK, so since u_{n_1} is ϵ_2 , this is still a bounded sequence by what I showed here, bounded sequence. This implies there exists a subsequence.

Now I'll call this subsequence u_{n_2} of $j \rightarrow \infty$. So in 2 of j , this is a sequence of integers. That is a subset of these guys that's increasing -- of u_{n_1} of j . So I should say-- which converges. So now note, this sequence converges. But the n_2 of j , so this is a subsequence of integers of the n_1 of j 's.

Since the n_2 of j 's a subsequence of the n_1 of j 's, we get that limit j goes to infinity of u_{n_1} of j -- or should say n_2 of j -- ϵ_1 . So this is a subsequence of this sequence. So the n_2 of j is just a subsequence of n_1 of j 's. So since this sequence in j converges, this subsequence also converges.

But now we've chosen the two so that we also have not just convergence along the first entry, but convergence along the second entry. But now you see what-- so this is how the argument goes, that now I will take a subsequence of the n_2 of j so that the subsequence converges when I pair it with ϵ_3 .

And then I take a subsequence of that subsequence so that I get ϵ_4 and so on, and so on. And all of these subsequences are subsequences of-- or these subsequences of integers are subsequences of the previous set of integers, which are again a subsequence of the original integers n .

So then what do we get? For all l there exists a subsequence of integers in l of j of the previous set of integers in, let's say, l minus 1 of j , such that for all k between 1 and l limit as j goes to infinity of u_{n_l} of j ϵ_k exists.

So I hope this is clear. So you just take a subsequence of the original sequence so that the first entry converges, take a subsequence of that so that the second entry converges, and so on and so on. So then you have this nested subsequences so that at each fixed l , you have convergence of the first l entries.

And so what you now do is that you pick-- so let me write or pick v_l to be the diagonal along this sequence. So u_{n_l} in l of l , where l equals $1, 2, 3$, and so on. Then what we get is since the l -th subsequence converges-- or the k -th entry for one-- OK, so let me back up and say this one more time.

Since for fixed l here I have convergence of the first l entries, by picking along the diagonal I get a subsequence of u_n such that for all k I have convergence for the k -th entry as l goes to infinity, converges.

All right, so I can take subsequences of subsequences and obtain a subsequence in the end so that for this subsequence of the original sequence u_n 's, I have convergence of the entries entry by entry, or I have convergence of the Fourier coefficients as l goes to infinity.

And now we're going to use this equi-small tails to prove that the sequence v_l converges. We're just actually going to show its Cauchy. Since H is a Hilbert space, we then conclude that it must converge. So claim v_l is Cauchy.

And then that will conclude the proof because then it must converge in H . And by what I said at the beginning, since this is a sequence of elements in k , which is a closed set, the limit has to be in k . All right, so we just need to show it's Cauchy, and then the proof is done.

So let ϵ be positive. So since k has equi-small tails-- let's go over here-- there exists natural number capital N so that for all l we have that the sum of squares k bigger than N of the tails, v_l, e_k squared, is less than ϵ squared over 16. Why 16? Because I want everything to come out correct in the end.

So since the N sequences given by the first N entries of the sequence v_l , so since the N sequences v_l, e_1 up to v_l, v_n, l -- so these are only-- so this is the bit where I said equi-small tails allows us to throw away the tail end of these entries, and just control over the first finitely many entries gives us the condition or the convergence that we want. This is where that's happening.

So since these converge, there exists a natural number M so that for all l, m bigger than capital M , I have that the sum n equals 1 to capital N of v_l, e_k minus v_m, e_k squared is less than ϵ squared over 4.

So each of these are convergent sequences. So each of these are Cauchy sequences. So for each one, I can find a capital $M_{sub 1}$ so that just one of these entries is very small for l and m bigger than that capital $M_{sub 1}$. And then I can do the next one so that for-- so this should be k . I don't know what that looks like, but k .

So that for k equals 2, it's very small, and then up to k equals n . And then I choose capital M to be the maximum over all those $M_{sub N}$'s. So I can always choose this because, again, there's only finitely many conditions I have to verify, all right.

So now I claim this capital M works. Then for all l, m bigger than or equal to M , if we look at the norm of v_l minus v_m , so this is equal to sum-- it should say k equals 1 to n of v_l minus v_m .

So by k equals-- k bigger than N , v_l minus v_m, e_k $1/2$. So here I'm using the fact that we have an orthonormal basis. Then in that case, the norm is equal to the sum of the squares of the Fourier coefficients.

Bessel's inequality would say this is less than or equal to this side. It wouldn't give us any control we want over the actual norm. But because we're working in orthonormal basis, the norm of that is equal to the sum of squares, take the square root.

And so now the square root of A plus B , that's always less than or equal to the square root of A plus the square root of B . So that's less than A equals 1 to N of v_l minus v_m squared $1/2$ plus k bigger than n v_l, e_k .

So I'm just going to write this out a little differently, $v_m, e_k, 1/2$. Now this part we know is less than-- by how we've chosen capital M , this finite part is small. It's less than ϵ over 2. So that's less than ϵ over 2.

Plus now we use the triangle inequality now for little l_2 . I can think of this as the sum in little l_2 -- or I can think of this as the norm in little l_2 of this sequence in k minus this sequence in k . So this is less than or equal to k bigger than $N, v_l, e_k, 1/2$ plus sum k bigger than $v_m, e_k, 1/2$

All right, we're almost done. So this is less than, the square part is less than ϵ^2 over 16. So taking the square root of that, I get ϵ over 4 by how we chose N . So this is based on how we chose N , plus-- which came from the equi-small tails part. And this part also because of N is less than ϵ over 4 equals ϵ . So we've now shown the claim that the subsequence v_{n_k} is Cauchy, and therefore it converges since H is complete.

Now, so for example, one can readily verify now that the following subset. Let K be the set of all sequences (a_k) with the property that $a_k \leq 2^{-k}$. So this is not a subspace, it's a subset-- is compact. This subset is known as the Hilbert cube.

All right, now maybe it's a bit unwieldy that our condition for being compact is phrased in terms of an orthonormal basis. Maybe that's not so simple to verify or definitely not necessarily, what's the word I'm looking for, canonical-- I guess that's the word I'm looking for-- in the sense that we have to make a choice to verify compactness.

But a different way of characterizing compact subsets of a Hilbert space is the following. Now, I'm not going to give the proof of this. You can look it up in Melrose's notes. Maybe I'll say a word about why you should believe or think it's true.

But, again, it involves a diagonal argument, which that was tough enough to do here, or at least painful enough to write out. And so I don't want to do it again. So we have the following that-- and this theorem also holds in a nonseparable Hilbert space, basically by a trick of reducing it to a separable Hilbert space, but that's OK.

A subset K of a Hilbert space H is compact if and only if K is closed, bounded. And the following condition holds what you can think of as K can be approximated by finite dimensional subspaces. And for all ϵ positive, there exists a finite dimensional subspace W in H such that for all u in K , if I look at the distance from u to this finite dimensional subspace-- which I think I said at one point, maybe I didn't, but a finite dimensional subspace is always a closed subspace in H .

A finite dimensional subspace is always closed. But if I look at the distance from this point to the subspace, this finite dimensional subspace, this is less than ϵ . So a subset in the Hilbert space is compact if and only if it's closed, bounded, and if you like can be approximated by compact subsets of finite dimensional subspaces, another way to think about it.

Now, why is this believable in the first place? Well, what this-- where is it? What the equi-small tail condition says is basically that all elements in K can be approximated by-- if you let ϵ be positive-- can be approximated by the subspace consisting of the span of the first one up to N basis vectors or orthonormal basis vectors. That's what this is.

For all ϵ the tails are small. Another way of saying that is that for all ϵ , you can approximate any element in the set K in the span of the first N basis vectors. So K has to be close to finite dimensional subspace.

But one can go the opposite direction as well, showing that finite dimensional subspace-- being able to be approximated by those guys implies that K is compact. And it's another diagonal argument that I just don't want to write out and rather just start using these conditions instead to start saying things about when we can solve certain equations and certain interesting operators that arise quite naturally.

So let's start talking about certain classes of operators. And we should start with the simplest, which are finite rank operator. So the only linear operators you came into this class knowing probably-- if you didn't and you were more sophisticated, that's good-- but were matrices.

Now a matrix is just a linear transformation defined in terms of an orthonormal-- or not orthonormal basis, but a basis in both the domain and target. We picked the target to be the domain. We just fix a basis, and then we can express a linear transformation as an array of numbers.

Finite rank operators are the generalization of what you think of matrices now for to Hilbert spaces. And we'll see that in just a minute. And what follows H is a Hilbert space. And instead of writing the space of bounded linear operators, $B(H, H)$, I'm just going to concatenate this by dropping one of those H 's. So the space of bounded linear operators from H to itself will be denoted by $B(H)$.

So finite rank operators, these are, like I said, these will be the analog of matrices. But let me give a invariant definition first. So a bounded linear operator is a finite rank operator if the range of T which is a subspace of H is finite dimensional, OK.

So, for example, let me give you the simple-- well of course the zero operator is a simple one. If H is a finite dimensional Hilbert space, [INAUDIBLE], then every bounded linear operator is a finite rank operator because the range is always contained in C_n , which is finite dimensional.

For example on little ℓ_2 , if I define T_n to be now the sequence given by a_1 over 1, a_2 over 2, up to an over n for some n and then 0 afterwards, and this is for a equals sequence in little ℓ_2 , then T is a finite rank operator because the range of this operator-- so here N is a fixed number acting on sequences.

And little ℓ_2 just spits out the first-- let's make this even more explicit. Let's say that's 5. This is a finite rank operator because it's contained in the subspace consisting of those sequences which are 0 after the fifth entry. And that's finite dimensional. A basis is given by the sequence consisting of 0's with a 1 here, 0's with a 1 here, 0's with a 1 here, and so on. So this is a finite rank operator.

And so let me just put over here that as far as notation goes, we write T is in $R(H)$, R for rank, finite rank. All right, so $R(H)$, this is the set of all finite rank operators. Now, it's not just a set. It's easy to see that this is a subspace of the Banach space consisting of all bounded linear operators from H to itself.

Why is that? Well, so if I take an operator that has finite dimensional range and multiply it by a scalar, then the range of the scalar multiple of that operator is equal to the range of the original operator unless it's the scalar 0. So that will be finite dimensional.

On the other hand, if I take two finite rank operators and I consider their sum, the range of the sum is going to be contained in the direct sum of the ranges. And the direct sum of two finite dimensional spaces, subspaces, is again finite dimensional.

So if one has dimension 5 and the other has dimension 6, then the direct sum of these two will be of dimension 11. So that is me talking you through the proof that this is a subspace. So take a minute to write it down. But now let me prove that these finite rank operators really are essentially matrices.

I have the following theorem characterizing finite rank operators. So T is a finite rank operator if and only if there exists an orthonormal, finite orthonormal subset e_k , k equals 1 to some integer L . And complex numbers c_{ij} i, j equals 1 to L .

So this is subsequence-- or a subset of complex numbers such that if I want to compute what T applied to u is, it's just going to be $\sum_{i=1}^L c_{ij} u_j e_i$, i, j goes from 1 to L of these numbers c_{ij} times the j entry of u times e_i . So, in other words, T corresponds to a matrix where the coefficients are these c 's, basically, acting on this finite dimensional subspace of e_k 's.

So let's give the proof of this. One direction is immediate. If T has such a representation, then it has finite rank because then the image is contained in the span of the e_i 's for i going from 1 to L . And that's a finite dimensional subspace. So this is clear that this condition implies that T is a finite rank operator.

So let's go the opposite direction. So since the range of T is finite dimensional, we can find an orthonormal basis, call it e_k , k equals 1 to N such that T applied to u is equal to-- so it's an element in the span of-- so this spans the range. So I can write this as some numbers times the e_k 's.

And what are these numbers? Well, these numbers, since these are orthonormal, have to be T applied to u in a product e_k . All right, so Tu -- so the ranges spanned by these orthonormal vectors, and anything in the span of this orthonormal vectors has to be written as that thing in a product e_k times e_k .

But let me now make an observation that this thing right here we can write as using the adjoint as u in a product $T^* e_k, e_k$. The adjoint operator, so T^* which goes from H to H , remember satisfies this property that given any two vectors u and v , T applied to u in a product v is equal to u in a product $T^* v$. That's the defining property of the adjoint

So I can write it like this. And-- which is k equals 1 to N of u in a product, call this $v_k, \bar{e}_k$, where here v_k is simply defined to be $T^* e_k$ now, let e_1 up to e_L be the orthonormal subset of H obtained by applying the Gram-Schmidt process to vectors E_1, E_2 up to E_L, v_1 up to v_L .

So applying it to this subset, I keep the e_1 up to e_L 's. And then I might pick up some more from some more orthonormal vectors that are the normal to these guys when I hit the v_1 's up to v_L . I should have gone to N , I'm sorry. So not L here, but N .

So I obtained an orthonormal subset that spans the same span of these guys. And so what does that mean? Then there exists numbers a_{kj}, b_{kj} -- let's call it i -- such that $\bar{e}_k$ is in the span of these orthonormal vectors. So I can write it as a_{ki}, e_i k equals 1 to L -- no, no, no, sums going the wrong way-- i equals 1 to L , and also the same for the v 's, some v_k equals j equals 1 to L b_{kj} .

So these are obtained by the Gram-Schmidt process applied to this collection of vectors. And therefore I can write each one of these vectors as a linear combination of these orthonormal vectors. So that's all I'm saying here. Now we just plug this in to this computation. And we have Tu which we computed was equal to $\sum_{k=1}^N u$ in a product $v_k, \bar{e}_k$.

This is equal to-- now if I stick in these sums here, I get $\sum_{i,j=1}^L \sum_{k=1}^N a_{ki} b_{kj} u_j e_i$, i, j goes from 1 to L of some k equals 1 to N of $a_{ki} b_{kj}$ complex conjugate. Now, u in a product e_j -- this is coming from when I stick in the v_k 's-- e_i . And this is coming from when I stick in what I have for the $\bar{e}_k$. And this number right here is my number c_{ij} .

And that's the end. So finite rank operators are really just matrices. You can just compute these numbers and they completely characterize the linear operator. So in particular, the kernel, the null space of T is the subspace that's orthogonal to the e_1 up to e_L , or at least includes that.

Now from this we can conclude a nice property about finite rank operators, namely the first is that if T is a finite rank operator, then its adjoint is also a finite rank operator. And if T is a finite rank operator and A and B are just bounded linear operators on H , then A times T times B , this is also a finite rank operator.

In fancy or in short form you would say that this is a star closed two-sided ideal in the space of bounded linear operators, or its closed undertaking adjoints in closed undertaking the two-sided multiplication by bounded linear operators.

So 2 I'll leave as an exercise. What's the point? The T has finite dimensional range. A then acts on that finite dimensional range. And therefore its range, the range of A hitting T must then be finite dimensional. And what you do with B really doesn't matter.

So 1 is an exercise. So let's prove 1. 1 we'll prove using this characterization of finite rank operators. So we're assuming T is finite rank. So we can write T as some constant c_{ij} , u and a product e_j , e_i , u in H . So we can express it in that way where these are just some fixed constants not depending on u .

Now let's compute how T^*x on vectors by using this defining property of the adjoint. Then u , T^*v -- what we're going to do is we're going to come up with an expression for this as u in a product something and from which we can conclude that T^*v has to equal that something for all v .

Now this is equal to, by the defining property of the adjoint, T applied to u , v . So this is equal to -- i and j will be understood to go from 1 to l . So this is just finite, a finite sum, OK -- c_{ij} , u , e_i -- or e_j , sorry -- e_i , inner product v .

Now taking the inner product is linear in the first entry. So this is equal to sum ij . Again, this is a finite sum ij , u , e_j , inner product e_i , v . And now I'm just going to switch what I'm summing. This I can write as now u inner product sum ij c_{ij} bar.

So if I want to -- this I can write as bar, bar, but e_i , v , complex conjugate, e_j . We can check this as correct because -- so u will then inner product with each of the e_j 's and these constants when they come out of the inner product, get hit by a complex conjugate, which turns them back into c_{ij} and e_i , v .

Now, the complex conjugate of the inner product is just the same as if I flip the entry. So this is equal to c_{ij} times v , e_i , e_j . Now, this holds for all u and v . So what have I shown? I've shown that u T^*v minus this finite sum, so I actually put it in here, c_{ij} v , e_i , v_j , this equals 0 for all u , for all v .

Now, for a fixed v this quantity here is orthogonal to everything in H . So it has to be 0. And therefore I conclude that T^*v has to equal sum from ij equals 1 to l , the complex conjugates of c_{ij} , v inner product e_i , e_j for all v in H . And that proves that T^* is a finite rank, and also gives you the expression for the coefficients in terms of the old ones.

I could write this-- now if I wanted to write it in terms of the i being in front of the bases vector that I'm having here, I write it as $\sum_j c_{ij}$. So we see that the matrix corresponding to the adjoint is what we computed last class basically, is the complex conjugate of the transpose of this matrix, of the original matrix c_{ij} 's. All right, so that proves adjoints are also finite rank.

Now, so taking the adjoint of a finite rank operator leaves you in the space of finite rank operators. If you take finite rank operators and compose them on the left and on the right with bounded operators, you remain finite rank.

The subset of finite rank operators also form a subspace in the Banach space of bounded linear operators on a Hilbert space. So the space of bounded linear operators on a Hilbert space naturally come with a norm as well, the operator norm. So the next obvious question is, is the subspace of finite rank operators, is this closed?

Closed not meaning-- not as a closed undertaking linear combinations. Of course that holds for every subspace. But if I take a sequence of finite rank operators that are converging in operator norm to something, then is the limit also a finite rank operator?

So that's the definition of being a closed subspace, or let me say subset so that we're now asking about this in a metric sense. Is it a closed subset in the space of bounded linear operators? Now, the answer to this is no. Why?

Let's, for example, take T_n . So this will be a sequence of operators from ℓ^2 to ℓ^2 . And for an element a in ℓ^2 , this should give me another sequence in ℓ^2 . And it'll be the sequence given by a_1 over 1, a_2 over 2, up to a_n over n , and then zeros after that.

All right, so T_1 , the operator T_1 takes the sequence in ℓ^2 and just spits out the sequence that has the first entry divided by 1. And 0's followed after that. That's the operator T_1 . T_2 takes a sequence in ℓ^2 and spits out the first entry divided by 1, the second entry divided by 2, and 0's after that, T_3 , so on.

Now, this is just a picture. This is formal. This is not meant to mean anything. If I were to express this as an infinite matrix times an infinite vector, or infinite length vector where the infinite length is the sequence a_k 's, this is $1, 1/2, 1/3, 1/n, 0, 0, 0, 0$.

This infinite matrix times a_1, a_2, a_3 , and so on. So I'm just multiplying a_1 by 1, a_2 by $1/2$, a_3 by $1/3$, and then up to a_n over n , multiplying a_n by $1/n$. And then the rest of the entries get spat out to give me 0. So these are the finite rank operators.

And you can check that the T_n minus T in operator norm goes to 0, where what is T ? T is the operator that takes a and spits out-- so a will be an element in ℓ^2 . Now as n goes to infinity, you can kind of guess what's going to happen is I end up multiplying everything by $1/n$ wherever entry I'm at, a_1 over 1, a_2 over 2, a_3 over 3, a_4 over 4, and so on, and on. So as an exercise you can show that the operator norm of T minus T_n , this is less than or equal to $1/n$, something like that, OK.

Now, this operator here does not have finite rank because then I can find infinitely many linearly independent vectors in the range. For example T of e_1 where this here is the basis vector gives me the-- this is the first basis vector in ℓ^2 . If this is the second basis vector in ℓ^2 , this is just-- and so on.

If I applied T to e_n , this is $\frac{1}{n}$, 0, 0, and so on. This is in the n spot. So it just takes each basis vector and-- so if you like, this is e_1 , this is $\frac{1}{2}e_2$, this is $\frac{1}{n}e_n$, and multiplies each of these basis vectors by $\frac{1}{n}$ over the integer that's-- not delineating-- denoting, I'm sorry.

All right, it's the end of a day so some of my thinking has slowed down. So finite rank operators are not closed-- is not a closed subset of the space of bounded linear operators. So what's the closure?

And finite rank operators we'd like to think we know how to solve equations for. I mean, just based on this expression for a finite rank operator, well, if we want to solve $Tu = v$, well, first we know v has to be in the range of or in the span of these basis vectors that appear here.

And now once we've restricted to that, then we can solve the equation by looking at the properties of this matrix. And the null space will be not only the null space that sits inside the span of these basis vectors, but it will also include the orthogonal complement in H of these basis vectors.

So solving equations involving finite rank operators ought to be fairly straightforward if we remember linear algebra. Unfortunately, they're not closed upon taking some subsequential limits. But we can identify the closure of-- so now the question is, what is the closure of finite rank operators in the space of bounded linear operators on H ?

So something in the closure has to be a limit of finite rank operators. And since we know how to solve finite-- equations involving finite rank operators, hopefully we know how to solve equations use with these operators as well. Is there a simple way we can characterize-- or not simple, but at least a more useful way to characterize them than just being the limit of finite rank operators?

So the answer to this question is-- so I won't even write-- OK, I'll write answer-- is what are called compact operators, so the subspace of compact operators. So let me make a definition. So k , a bounded linear operator is compact operator if the closure of the image of the closed unit ball in H -- so the closure of the image by k -- or the closure of the image of the closed unit ball is compact. I can also write this as a closure of the subset $\{ku : \|u\| \leq 1\}$.

So I won't have time to prove this theorem in this class, but we'll do it next time. And we'll actually show that, in fact, this class of compact operators which are defined in this way, this is a compact operator according to this definition if and only if k is in the closure of finite rank operators, meaning there exists a sequence of finite rank operators converging in operator norm to k .

And, again, why are we interested in compact operators? Well, we know how to solve finite rank or equations involving finite rank operators. We just look at the subspace which that contains the range. And then we look at the matrix that appears there in that decomposition.

Compact operators there are something completely new. When we move from finite dimensional linear algebra now to functional analysis because these are not exactly necessarily equal to finite rank operators. I mean, we just did this example a minute ago of this operator given here by where I take the entry of the sequence and divide it by the place that it occurred in the sequence for a sequence given in ℓ_2 .

So if you believe the theorem, this is a finite rank-- I mean, not finite. This is a compact operator. It's the limit of this sequence of finite rank operators. Perhaps you want to take a try at trying to prove this as a compact operator directly from the definition, or not. you can wait till next time and just use this theorem.

But many other operators, many interesting operators, for example, inverses of differentiable operators in fact turn out to be compact operators. That's also why we're interested in them. So we got these by taking the closure of finite rank operators.

And then after-- so now we have this interesting subspace of operators that we hope to be able to solve equations involving. And that'll be basically what takes up the remainder of our time in this course.

So you just staring at this definition, you can kind of see that-- well, you can see that if k is a finite rank operator, then it's a compact operator. So we're not going to prove this theorem, like I said, but just to see or do a sanity check that finite rank operators ought to be compact operators.

Well, a finite rank operator, the image of this closed unit ball of a finite by a finite rank operator, that's going to be a bounded subset of a finite dimensional subspace. Any bounded operator takes bounded sets to bounded sets. So if it's finite rank, it takes this bounded set to a bounded subset of a finite dimensional subspace of H .

And then I take the closure. I get a bounded and closed subset of a finite dimensional subspace. And we know by Heine-Borel that closed and bounded subsets of finite dimensional subspaces are compact. So that shows why at the very minimum you should believe that finite rank operators are compact. OK, so we'll stop there.