

[SQUEAKING]

[RUSTLING]

[CLICKING]

**PROFESSOR:** OK, so we're going to continue with our discussion of the spectrum of an operator, a bounded linear operator. So let me just recall from last time the definition that if  $A$ -- and throughout  $H$  is a Hilbert space, if  $A$  is a bounded linear operator, then the resolvent set of  $A$  is the set of all  $\lambda$  and complex numbers such that  $A$  minus  $\lambda$  times the identity, which I write as  $A - \lambda I$ , is bijective, meaning it's 1 to 1 and onto, which implies by the open mapping theorem, that it has a bounded inverse or is equivalent to it having a bounded inverse. And the spectrum of  $A$  was the complement of the resolvent set within the set of complex numbers.

And so the spectrum is supposed to be a generalization of, in finite dimensions, what were called the eigenvalues. And so we just-- recall we called  $\lambda$  an element of the spectrum, an eigenvalue if there exists a  $u$  not equal to 0 such that  $A - \lambda I$  applied to  $u$  is 0. In other words,  $A - \lambda I$  is not injective. So the reason for  $\lambda$  being in the spectrum is that-- so for a number to be called an eigenvalue is that this operator,  $A - \lambda I$ , has non-trivial null space.

In other words, it has a non-zero  $u$  so that  $Au = \lambda u$ . And we call this thing an eigenvector. And we saw last time examples of an operator that has infinitely many eigenvalues and eigenvectors and also an example of a bounded linear operator, which has no eigenvalues and eigenvectors unlike in the case in finite dimensions, where the spectrum is exactly the set of eigenvalues of an operator.

Now, and we also proved at the end of last time-- so what we could say about these two sets or what we could say about the spectrum is that it's a closed set. And it's contained within the ball of radius  $\|A\|$  in the complex numbers, which means it's a compact set. And what we could say, by taking complements about the resolvent set, is that it's an open set that contains the exterior to a ball of radius  $\|A\|$  in the complex numbers.

And that's about all we can say about the spectrum in general for now. But we can say quite a lot about the spectrum of self-adjoint operators. And then we can give a pretty complete picture about the spectrum for compact operators-- self-adjoint compact operators. But let's first look at self-adjoint operators.

So at the end of last time, we proved that if I have a self-adjoint-- and this is not related to the spectrum. If I have a self-adjoint bounded linear operator on a Hilbert space, then for all  $u$ ,  $\langle Au, u \rangle$  is a real number. And we could write the norm of  $A$  as this quantity  $\sup_u \frac{\|Au\|}{\|u\|}$  equals  $\sqrt{\sup_u \langle Au, u \rangle}$  of the absolute value of  $\langle Au, u \rangle$ .

All right, so now, we have the following theorem about the spectrum for self-adjoint bounded linear operators on a Hilbert space. So the first is that the spectrum is contained in the real number line. So the spectrum of  $A$  is contained in the real number line--  $[-\|A\|, \|A\|]$  or in this interval, minus  $\|A\|$  to  $\|A\|$ , which I'm viewing as a subset of the complex numbers, so just the line segment from minus  $\|A\|$  to  $\|A\|$  as a subset of the complex numbers. And the second is that one of these two endpoints has to be in the spectrum-- maybe both, but at least one. At least one of  $\pm \|A\|$  is in the spectrum of  $A$ .

OK, so to establish one, we already know-- since the spectrum of  $A$  is contained in those complex numbers with modulus less than or equal to the norm of  $A$ , we just need to show that the spectrum is contained in the real number line. Then it must be contained in this interval, since it's contained here.

OK, so we'll show that anything off the real number line lies in the resolvent. That's how we'll go about this. So we'll show that if  $\lambda = s + it$  with  $t \neq 0$ , then  $\lambda$  is in the resolvent set of  $A$ .

Now, suppose  $\lambda$  has this form. Then  $A - \lambda$  is equal to  $A - s - it$ , or minus, sorry,  $-it$ , which I can write as  $\tilde{A} - it$  with  $\tilde{A}$ , a bounded linear operator given by  $A - s$ , which is also equal to the adjoint because  $s$  is a real number. So I should have said  $t \neq 0$  and  $s, t$ , real numbers.

So  $A - \lambda$  I can write as  $\tilde{A} - it$ , where  $\tilde{A}$  is  $A - s$ , again, a self-adjoint operator, all right? So if I can do an argument for  $\tilde{A}$  and show that  $\tilde{A} - it$  is bijective, then I can conclude  $A - \lambda$  is bijective. Why am I doing this? Because then I can just focus on the one case that  $s = 0$ .  $A - it$  is bijective if and only if  $A - \lambda$  is bijective.

So we only need to consider-- so I can just work on this thing. But instead of writing  $\tilde{A}$  over and over again, I'll just switch back to  $A$ . So I only really need to consider the case  $s = 0$ . So rather than do the argument for  $\tilde{A} - it$ , I'm just going to set  $s = 0$  and start doing the argument for  $A - it$ .

OK, so since by what we proved, or the result from last time-- so let me just set out what we're going to prove. If  $A$  is self-adjoint, then  $A - it$  is bijective for all  $t \neq 0$ . So once I've proven this claim, then I've proven that-- I've proven my first part of the theorem.

OK, now, since  $Au$  applied to  $u$  is real, I get that-- if I take the imaginary part of  $A - it$  applied to  $u$ , inner product  $u$ , this is equal to--  $\langle Au, u \rangle$ , taking the imaginary part of that is just 0. So then I get  $-t \|u\|^2$ , which implies that since  $t$  is non-zero-- we're assuming  $t$  is non-zero--  $A - it$  times  $u = 0$  if and only if  $u = 0$  because if this quantity here equals 0, then this thing here equals 0. And therefore, the norm of  $u$  has to be 0 since  $t$  is non-zero. So the null space of  $A - it$  is just the zero vector. And therefore, it must-- it's injective, right?

It is injective. It is 1 to 1. All right, now, we just want to show it's bijective-- or it's surjective. OK, so similarly, I can prove that the adjoint of this operator, which is, in fact,  $A + it$ , is injective. And therefore, I get that the orthogonal complement to the range of  $A - it$ -- so I want to show this equals  $H$  to show it's surjective. So the orthogonal complement of that, which is equal to the null space of the adjoint, which is-- equals-- so since this equals 0, I conclude that the range of  $A - it$  closure, which is equal to the range of  $A - it$  taking the orthogonal complement of the orthogonal complement.

So that was part of an assignment that if I have a subspace of a Hilbert space and I take-- or let me say it here-- and I take the orthogonal complement of the orthogonal complement, I don't get back to subspace. I get the closure of the subspace. So this is equal to the orthogonal complement of the zero vector, which is  $H$ . So I'll be done showing that  $A - it$  is surjective if I can show that the range is closed because then this will just be the range of  $A - it$  equals  $H$ .

So we just need to show now that the range of  $A - it$  is closed. So to show it's closed, we have to show that if we take a sequence of elements in here converging to something, then that limit is, in fact, in the set. So suppose I have a sequence of elements  $u_n$  such that  $A - it$  applied to  $u_n$  converges to an element  $v$ . So my goal is to show that  $v$  is in the range of  $A - it$ .

So we want to show  $v$  is in the range. And then we've shown that the range is closed. And we're done with the first part.

So using this argument here, we're going to show that the  $u_n$ 's, which a priori we don't know converge-- all we know is that the images of the  $u_n$ 's converge. We are going to show that the  $u_n$ 's actually converge. And then that will essentially finish the proof.

Then we have that the absolute value of  $t u_n - u_m$  norm squared-- this is, by this calculation we did over here, equal to the absolute value of the image of-- I mean, the imaginary part of  $A - it$  applied to  $u_n - u_m$ . Take the value of all of that.

And now, this is less than or equal to-- so the absolute value of the imaginary part of a complex number is less than or equal to the absolute value of that complex number, which by Cauchy-Schwarz I can say is less than or equal to  $A - it$  applied to  $u_n - u_m$ . But I'll write it as  $A - it$  times the norm of  $u_n - u_m$ .

And I started off with  $t$ , which is non-zero, times the norm of  $u_n - u_m$  squared. So I get that  $u_n - u_m$ -- that this is less than or equal to  $1$  over the absolute value of  $t$  times the norm  $A - it$  applied to  $u_n - u_m$  norm.

Now, this thing on the right-- or I should say  $A - it$  applied to  $u_n$ , this is a convergent sequence. In particular, it's a Cauchy sequence. So given  $\epsilon$ , I can find capital  $N$  so that the norm of this right-hand side is less than  $\epsilon$  times the magnitude of  $t$ . And therefore, for all capital  $N$  bigger than or equal to-- or for all little  $n, m$  bigger than or equal to that capital  $N$ , this in norm will be less than  $\epsilon$ .

Since this is Cauchy because it's convergent, the previous estimate implies that the sequence  $u_n$  is Cauchy. And since we're in a Hilbert space, which means it's complete, we can find a limit of this  $u_n$ . There exists a  $u$  in  $H$  such that  $u_n$  converges to  $u$ . And then we're done now.

Then since  $A$  is a bounded linear operator,  $A - it u$ -- or  $A - it$  applied to  $u$ , this is equal to the limit as  $n$  goes to infinity of  $A - it$  applied to  $u_n$ . But remember, we assumed that this converges to some element  $v$ . And therefore,  $v$  is equal to something in the image or in the range of  $A - it$ .  $v$  is in the range.

And thus, the range of  $A - it$  is closed. And by this here, we conclude that the range of  $A - it$  equals  $H$ . So  $A - it$  is bijective. And that concludes the proof of the first property we wanted to do.

OK, so for the second thing we wanted to show, we wanted to show that at least one of  $\pm \|A\|$  is in the spectrum of  $A$ . Now, since the norm of  $A$  is equal to the sup over norm of  $u$  equals  $1$  of the absolute value of  $Au$ ,  $\|A\|$ -- so a supremum is always characterized by being an upper bound and also there existing a sequence in the set of things you're taking the supremum of converging to that supremum.

So that implies that there exists a sequence of unit vectors so that  $\|Au\|$  inner product  $u$  in absolute value has to converge to the norm of  $a$ . So in particular,  $\|Au\|$  applied to  $u$  has to converge to the norm of  $A$  or minus norm of  $A$  as  $n$  goes to infinity. And there's no  $n$  here.

All right, then what does this imply? Then this implies that, for at least one of these choices, then  $A$  plus or minus norm of  $A$  applied to  $u$  inner product  $u$  converges to 0 as  $n$  goes to infinity, where here the plus or minus is chosen depending on whether this goes to plus or minus the norm of  $A$ . So this sign here would be the opposite of whichever sign this sequence goes to.

I now claim that this property here implies that whichever sign we have that for, that this operator appearing here cannot be invertible. And therefore, whichever sign appeared here or the opposite sign-- so let me, in fact, stay-- so the minus corresponds to the plus sign. The plus sign corresponds to the minus sign if we have one of those. So I claim that this property here implies that this operator is not invertible. And therefore, one of those is in the spectrum.

So suppose instead that  $A$  minus plus norm of  $a$ , whichever one appeared, is invertible. So whichever one does satisfy this is invertible. Then the  $u_n$ 's all have norm 1. So 1 is equal to the norm of  $u_n$ . And I can write this as the norm of  $A$  minus plus norm  $A$  inverse applied to  $A$  minus plus norm  $A$ , because that's just the identity, applied to  $u_n$ .

And this is less than or equal to the norm of the inverse times the norm of this quantity. And so this is a fixed number. And this thing is converging to 0. So the right-hand side converges to zero. But that's 1, right? I have 1 is less than or equal to 0. So that's a contradiction. Thus,  $A$  minus or plus the norm of  $A$ -- again, the minus or plus corresponds to which sign of the norm of  $A$  we had that sequence converging to-- is not invertible, which implies that plus or minus at least one of these is in the spectrum of  $A$ .

OK, now, we can, in fact, do a little bit better than-- based on this argument, we can do a little bit better in bounding the spectrum of a self-adjoint operator than just the bound that we have coming from the general theory. So what do I mean by that?

So if  $A$  is a self-adjoint bounded operator and  $a_{\min}$  is equal to the infimum over all  $u$  equals 1  $\|Au\|$  applied to  $u$ ,  $a_{\max}$  equals the sup  $\|Au\|$  applied to  $u$ , then the spectrum of  $A$  is contained. So first off then, a couple of things, two things-- then both of these numbers are in the spectrum of  $A$ . And the spectrum is contained in this line segment.

So this is something of a tighter bound because  $a_{\min}$  is always bigger than or equal to minus the norm of  $A$  just by this always being bounded below by the norm of  $A$ . And  $a_{\max}$  is always bounded above by the norm of  $A$ , since this is always bounded above by the norm of  $A$ . So the sup will be bounded above by that. So this is a tighter estimate than just the regular estimate that says the spectrum is contained inside of the interval from minus norm  $A$  to norm  $A$ . And in fact, you get more information, that not just one of the endpoints have to be in, but both of these endpoints are in the spectrum.

So the proof of this is just kind of a trick of using what we've done already. So first, note that-- again, since the absolute value of  $\langle Au, u \rangle$  is always less than or equal to the norm of  $A$ , for all unit vectors, this implies that this quantity here is always bounded below by  $- \|A\|$  and bounded above by  $\|A\|$ . And therefore, the infimum of this is always bounded from-- so this is a lower bound for this quantity here. So this infimum is bigger than it, since it's the greatest lower bound. And the least upper bound of these quantities is always less than or equal to the norm of  $A$ . So these are actual numbers for one.

OK, now, by the definition of  $\alpha_+$  or  $\alpha_-$ , there exists sequences of unit vectors  $u_n$  or  $v_n$  such that  $A$  applied to  $u_n$  or  $v_n$  inner product  $u_n$  or  $v_n$  converges to  $\alpha_+$  or  $\alpha_-$ . Now, by the argument we just gave with  $\alpha_+$  or  $\alpha_-$  being the norm of  $A$ -- but now, we have this property, i.e.  $A$  minus  $\alpha_+$  or  $\alpha_-$  applied to  $u_n$  or  $v_n$ ,  $u_n$  or  $v_n$  converges to 0.

Since I have this property by the previous argument I gave, this implies that both  $\alpha_+$  and  $\alpha_-$  are in the spectrum of  $A$ , since we have for each choice of plus or minus a sequence of unit vectors so that this quantity here goes to 0. A minute ago, we could just assert that there was a sequence of unit vectors. So for at least one choice of plus or minus the norm of  $A$ , we had this thing going to 0. But for these two numbers, because it's the inf and because this is the sup, we can always find unit vectors so that this quantity is converging to the sup, which is  $\alpha_+$ ; this quantity is converging to the inf, which is  $\alpha_-$ .

So by the previous argument, we get that both  $\alpha_+$  or  $\alpha_-$  are in the spectrum of  $A$ . Again, I just want to emphasize. Before, we could just say that one of the norms of  $A$  or plus or-- at least one choice of plus or minus the norm of  $A$  is in the spectrum. Here, we're saying both of these numbers are in the spectrum. So now, what remains is to show that the spectrum is, in fact, contained in this interval from  $\alpha_-$  to  $\alpha_+$ .

All right, so let  $b$  be their midpoint. And  $B$  equals  $A$  minus  $b$  times  $I$ . Now,  $B$ 's a real number because those are two real numbers. So capital  $B$  is the difference between  $A$  and-- is  $A$  minus a real number times the identity. So  $B$  is self-adjoint and a bounded linear operator on  $H$ .

So by the previous theorem, we get that the spectrum of  $B$ , well, is contained in the norm of  $B$ -- so minus norm of  $B$ , norm of  $B$ . And it shouldn't take much thought to realize that if the spectrum of  $B$ , which is a shift of  $\alpha$  by little  $b$ , is contained in this interval, then the spec of  $A$  is contained in minus norm of  $B$  plus little  $b$ , norm of  $B$  plus  $b$ .

So now, what's left is to compute the norm  $B$ , all right? But this is not too difficult. We have that the norm of  $B$ -- this is equal to the sup of  $\|Bu\|$  applied to  $u$ . And now, I take the sup over all  $u$  equals 1. And let me plug in what  $A$  is and  $B$  is. And this is  $\|Au - \alpha_+ u\|$  or  $\|Au - \alpha_- u\|$  over 2.

Now, here's the picture. Here's  $\alpha_-$ . Here's  $\alpha_+$ .  $\alpha_+$  is the sup over all of these expressions where  $u$  has unit length.  $\alpha_-$  is the inf over all these expressions, where  $u$  has unit length.  $\alpha_+$  plus  $\alpha_-$  is the point right in the middle of them.

So what's the biggest this-- or what's the supremum of the difference between these numbers and the midpoint? Well, it's the distance given by the distance from  $\alpha_+$  to the midpoint, which is equal to the distance from  $\alpha_-$  to the midpoint, which is  $\alpha_+ - \alpha_-$  over 2. And since that's the norm of  $B$ , when we plug that into what we had a minute ago, we conclude that the spectrum is contained in  $\alpha_-$ ,  $\alpha_+$ .

So as a simple corollary of what we've done, we have this nice little statement about when exactly a self-adjoint bounded linear operator is non-negative. So let  $A$  be a self-adjoint bounded linear operator on a Hilbert space. Then for all  $u$ ,  $\langle Au, u \rangle$  is bigger than or equal to 0 if and only if the spectrum of  $A$  is contained in the non-negative numbers.

So I'm not even going to write out the proof. I'm just going to talk my way through it. So let's suppose that  $\langle Au, u \rangle$  is non-negative. Then this number  $a_{\min}$  is non-negative. And therefore, the spectrum is contained in  $a_{\min}, a_{\max}$ , which is the subset of the non-negative real numbers.

On the other hand, suppose that the spectrum of  $A$  is contained in here. Then  $a_{\min}$ , which is in the spectrum, has to be in the set of non-negative real numbers. And therefore,  $\langle Au, u \rangle$  always has to be non-negative, since  $a_{\min}$  is the inf over all of these.

So now, we're going to move on to the spectral theory for not just self-adjoint operators, but self-adjoint operators that are also compact. Again, a natural example is given by the inverse of taking the second derivative along with requiring 0 at the endpoints, this operator I gave last time. That is a bounded self-adjoint-- or a compact self-adjoint operator.

So all the spectral theory we developed for that applies. And the spectrum for that operator ends up being  $1/u^2$  over the eigenvalues corresponding to  $u'' = -\lambda u$  with 0 at the endpoint. And you'll see that in the assignment. Or maybe I'll do it as an example.

So now, we're moving on to spectral theory for compact self-adjoint operators, which is one of the most, again, complete things-- or class of operators we can say the most about when it comes to the spectrum. And I'll go ahead and give you a preview of what we can say about the spectrum for these operators, that it essentially consists of nothing but eigenvalues with the possible exception of 0 being an accumulation point of the eigenvalues.

So what we'll prove is that the spectrum of a compact self-adjoint operator consists of the eigenvalues of this operator along with possibly 0. And 0 may or may not be an eigenvalue. If it's not an eigenvalue, then it's the limit of the eigenvalues. And in fact, implicit in that statement is that the spectrum is, in fact, countable for a compact self-adjoint operator.

So why should we expect that? Or why should we expect such a complete picture? In the end, we'll also prove that you can find a basis for  $H$  consisting entirely of eigenvectors of the operator  $A$ , which is, again, a generalization to infinite dimensions of what hopefully you saw in finite dimensions. But if you didn't, our proof will still apply to finite dimensions.

So why should that then apply to compact self-adjoint operators if you believe it for finite dimensions? Well, it's because, again, compact self-adjoint operators are the norm limit of finite rank operators, all right? And finite rank operators, again, these just correspond to basically matrices. We know how to compute the eigenvalues of matrices. For finite rank operators, they could have a very large null space, meaning the eigenvalue 0 could have a very large eigenspace. But that's the point of why you expect maybe things to carry over to the setting of compact self-adjoint operators from what you know in finite dimensions.

OK, so this is not so much a definition as just notation. If  $A$  is a bounded linear operator, I will denote  $E_\lambda$  to be the null space of  $A - \lambda I$ -- in other words, the set of-- or the subspace of eigenvectors with eigenvalue  $\lambda$ , which, again, is the set of  $u$  in  $H$  such that  $(A - \lambda I)u = 0$ .

So first off, before we get to classifying the spectral-- or the spectrum of a compact self-adjoint operator as basically consisting of eigenvalues along with 0, we'll first give some kind of general properties of eigenvalues in general for a compact self-adjoint operator. So we have the following theorem that suppose  $A$  is a compact self-adjoint operator.

Then a few things-- if  $\lambda \neq 0$  is an eigenvalue of  $A$ , then the dimension of  $E_\lambda$ , the eigenspace, the linear subspace of all vectors that are eigenvectors of  $A$ , this is finite. So for a given eigenvalue, the dimension of the eigenspace is finite.

The second is that if I take two different eigenvalues, the corresponding eigenspaces are perpendicular to each other.  $\lambda_1 \neq \lambda_2$ . For eigenvalues of  $A$ , then  $E_{\lambda_1}$ ,  $E_{\lambda_2}$  are orthogonal or perpendicular. Every element in  $E_{\lambda_1}$  is orthogonal to every element in  $E_{\lambda_2}$  and vice versa.

And finally, the set of non-zero eigenvalues of  $A$  is either finite or countable. If it is countable, i.e. it's given by a sequence  $\lambda_n$ , then these-- or if it is countably infinite, I should say-- and I should have said countably infinite here. If it's countably infinite, then the eigenvalues converge to 0.

In particular, this implies that if I have a compact self-adjoint operator with infinitely many eigenvalues, then 0 is in the spectrum of this operator because the spectrum is a closed set. So it's closed under taking limits. And since, these are in the spectrum, the limit has to be in the spectrum.

All right, so proof of 1-- suppose I have a non-zero eigenvalue. And towards the contradiction,  $E_\lambda$  is not finite dimensional. Then what I can do-- by the Gram-Schmidt process, then there exists a sequence or a countable collection  $u_n$ , orthonormal elements in  $E_\lambda$ . So every element in the sequence has unit length. And it's orthogonal to any other element in the sequence.

Now, since  $A$  is a compact operator and all of these have unit length, it follows that  $Au_n$  is contained-- this is a sequence in a compact set, right? So it has a convergent subsequence--  $Au_{n_j}$ ,  $j$ . Then  $Au_{n_j}$  is Cauchy. But let's actually look at what's the difference between two of these in norm.

Let's make it squared. This is equal to norm of, because these are eigenvalues,  $\lambda u_{n_j} - \lambda u_{n_k}$  squared, which equals-- squared, which equals  $2\lambda^2$ , which is a fixed number that's positive because  $\lambda \neq 0$ .

Oh, I left off a part of the-- OK, so what does this imply? This implies that the distance between any-- so this is-- if I take any two elements in this subsequence, their distance is a constant equal to  $2\lambda^2$ . And therefore, this is not Cauchy, which is a contradiction.

Something I forgot to say-- restatement of the theorem-- forgive me, it's the end of a long day-- is that eigenvalues have to be real for self-adjoint compact operators, or really for self-adjoint operators. So I could have included it earlier. So the eigenvalues of a self-adjoint operator have to be real. Why is that?

Since if I have something with norm 1-- so if  $\lambda$  is an eigenvalue, it comes with an eigenvector  $u$  with length 1 so that  $Au$  equals  $\lambda u$ . Of course, it just has to be a non-zero  $u$ . But I can normalize it by dividing by its length.

And therefore, I get that  $\lambda$ , which is equal to  $\lambda u$  inner product  $u$ -- this is norm of  $u$  squared, which is 1  $\lambda u, u$ . And this is equal to complex conjugate-- or let's not do that. This is equal to  $Au, u$  which is equal to-- I take  $A$  and it becomes  $A^*$   $u, u$ . But  $A^*$  is equal to  $A$ , So I get  $u, Au$ , since  $A$  is self-adjoint. And this is equal to  $u, \lambda u$ .

And remember, inner products are conjugate linear in the second entry. So this  $\lambda$  pops out, but now complex conjugate. So  $\lambda$ -- so we've shown that the complex conjugate is equal to the original number. So  $\lambda$  has to be a real number.

All, right so that proves part 1, that the eigenvalues of a self-adjoint operator have to be real. And the eigenspaces, which is what I have just started calling the  $E_\lambda$ s, the eigenspaces, have to be finite dimensional for a compact self-adjoint operator. OK, so now, let's show that distinct eigenspaces have to be orthogonal to each other. Suppose  $\lambda_1$  does not equal  $\lambda_2$ .  $u_1$  is in  $E_{\lambda_1}$ .  $u_2$  is in  $E_{\lambda_2}$ .

So now, what I'd like to show is that the inner product of  $u_1$  with  $u_2$  is equal to 0. And it's going to be a trick, kind of like I just did here.  $\lambda_1$  times  $u_1, u_2$ , this is equal to  $\lambda_1 u_1, u_2$ . This is equal to  $A$  applied to  $u_1, u_2$ . And now, I move  $A$  over to here because  $A$  is self-adjoint.

And  $A$  applied to  $u_2$ -- so  $u_2$  is in the second eigenspace. So this is equal to  $u_1 \lambda_2 u_2$ . And because  $\lambda_1$  and  $\lambda_2$  have to be real numbers-- what we've done from the first part-- this  $\lambda_2$  comes all the way out and remains itself, no complex conjugate because it's equal to its complex conjugate. And so I started off with  $\lambda_1$  times the inner product of  $u_1, u_2$ . And I've ended up with  $\lambda_2 u_1$  inner product with  $u_2$ . And therefore,  $\lambda_1$  minus  $\lambda_2$  times the inner product of  $u_1$  minus-- or the inner product of  $u_1$  with  $u_2$  equals 0.

And  $\lambda_1$ -- remember, we're assuming  $\lambda_1$  and  $\lambda_2$  are non-zero-- or not equal. So this quantity here is non-zero. So I get that  $u_1, u_2$  equals 0. And that's the-- nope, that's not the end. That's the end of number 2, but not the end of the proof of this theorem.

All right, so we're going to prove the last thing, that the set of non-zero eigenvalues is either finite or countable, and that if I arrange them in a sequence, then the sequence converges to 0. OK, so just to have some notation running around-- capital  $\lambda$ , let me let this denote those non-zero eigenvalues.

All right, so what I'd like to claim-- or what I'm going to show is that if  $\lambda_n$  is a sequence of distinct elements-- or distinct eigenvalues, non-zero eigenvalues of  $A$ , then these converge to 0. So this gives us-- of course, so the set of non-zero eigenvalues may be finite. Fine. Suppose it's not, OK? Now, we're just in the setting that  $A$  has infinitely many eigenvalues.

If I can prove this claim, then I have proven two things at once. I have proven both that the set of non-zero eigenvalues is countably infinite, assuming it's infinite, and they converge to zero. So why does this--



So first off, if we can show that this capital lambda is countable, then this claim then implies that-- or countably infinite, then this claim tells me that the eigenvalues converge to 0, which is the last thing I want. So all I really need to show is that this is countable using this claim.

Now, why does this show that capital lambda is countable? Since then if I define lambda sub capital N to be the set of non-zero eigenvalues which are, let's say, even bigger than or equal to  $1/N$ , this has to be a finite set, right? If it was infinite, then I could find a sequence of distinct elements in here and obtain-- or I should say, then I can find a subsequence-- or hold on. Let me stop for a minute.

So my claim is that this is finite for all N, which implies that lambda, which is the union of-- is countable. OK, so assuming this claim or assuming what I wrote here, that this is finite for all N, this implies this is countable, that's clear. So why do I get this as finite, this set is finite assuming this claim?

Well, if this set is infinite, then I can pick out a sequence of distinct elements in lambda sub N that converges. I could just take any sequence, and then take a convergent subsequence because that sequence has to be bounded between  $1/N$  and the norm of A. But since they're all bigger than or equal to  $1/N$ , that sequence has to converge to something that's non-zero. But that would contradict the claim-- again, assuming the claim is true. We haven't proved it yet, all right?

So again, from this claim, we can then conclude that each of these sets is finite for all N. And therefore, the set of non-zero eigenvalues is countable. And if it's countably infinite, then, again, from this claim, we conclude that the eigenvalues must converge to 0 when I line them up in a sequence. So the whole proof is reduced to just proving this claim.

OK, so to prove the claim, let  $u_n$  be associated eigenvectors. So these have unit lengths. And for all n,  $A u_n$  equals  $\lambda_n u_n$ , right? We have eigenvalues. So we can find eigenvectors with unit length.

Now, then  $\lambda_n$ , which is equal to-- or the absolute value of  $\lambda_n$  is equal to the absolute value of  $\lambda_n$ -- or the norm of  $\lambda_n$  applied to-- or times  $u_n$ , which is equal to the norm of A applied to  $u_n$ . So what I'm going to show is that A applied to  $u_n$  converges to 0. So if you like, this is the final claim that I need to. Prove so this is claim 1.

Claim 1 will follow from claim 2 in this little computation right here, where claim 2 is that the norm of A applied to  $u_n$ -- again,  $u_n$ 's are eigenvectors with unit length corresponding to the  $\lambda_n$ 's converge to 0. So the fact that A applied to these unit vectors converges to 0 is not just specific to eigenvectors of distinct eigenvalues.

It's just a property of the compactness of A and the fact that the  $u_n$ 's are in orthonormal sequence. They're all unit length. And any one element in the sequence is orthogonal to a different element in the sequence.

So suppose not. Suppose claim 2 does not hold. Then just negating the definition of convergence, there exists an epsilon positive. And we can find a subsequence  $A u_{n_j}$  such that for all j, length of  $A u_{n_j}$  is bigger than or equal to epsilon 0.

If you look at the definition of-- or the definition of convergence to 0 and then negate that, you can conclude that you can find a subsequence so that I have this. So there is some bad epsilon 0 so that I have that.

All right, since  $A$  is a compact operator, there exists a further subsequence. And let me call it  $e_{k_j}$ , which is  $u_{k_j}$  such that-- so remember,  $A$  applied to  $u_{k_j}$  so  $u_{k_j}$  is a unit length vector. And therefore,  $A$  applied to that is contained in a compact set, assuming  $A$  is a compact operator. So this must have a convergent subsequence such that  $A$  applied to  $e_{k_j}$  converges in  $H$ . And note  $\|Ae_{k_j}\|$ , since this is just a subsequence of this sequence, is bigger than or equal to  $\epsilon/2$  for all  $k_j$ .

Now, since the  $e_{k_j}$ 's are a subsequence of an orthonormal sequence, it's still an orthonormal sequence. So note, for all  $k_j \neq l$  inner product  $\langle e_{k_j}, e_l \rangle$ , which is  $\langle u_{k_j}, u_l \rangle$  equals 0. And what I'm using here-- so of course, these are all unit vectors. Why are they orthogonal? It's because they correspond to distinct eigenvalues, distinct non-zero eigenvalues.

And we proved that-- that was number 2, that-- was it number 2? Yeah, that eigenvectors for distinct eigenvalues are orthogonal to each other. So assuming the negation of the claim 2, which would prove claim 1 and finish the proof of this whole theorem, I conclude that there exists a sequence of eigenvectors, orthonormal eigenvectors of  $A$  so that  $\|Ae_{k_j}\|$  is always bounded below in norm by  $\epsilon/2$ .

And this sequence converges. So let  $f$  be the limit as  $k_j$  goes to infinity of  $Ae_{k_j}$ . Then norm of  $f$ , by continuity of the norm, is equal to the limit of the norms of the  $e_{k_j}$ 's. And all of these are above-- bigger than or equal to  $\epsilon/2$ . So  $f$  is non-zero, right?

In fact, we can say a little bit more. Then in fact-- let's see. So this is kind of useless information I skipped. I didn't write down what I wanted to. But then-- well no, I still need that. No, let me not get rid of that. So that should still be there.

So the norm of  $f$  is bigger than or equal to  $\epsilon/2$ . So norm of  $f$  squared is bigger than or equal to  $\epsilon^2/4$ . So  $\langle f, f \rangle$  and by continuity of the inner product, that's equal to-- since the  $Ae_{k_j}$ 's converge to  $f$ , I will get  $f$  here. And using the fact that  $A$  is self-adjoint, this is equal to  $\langle e_{k_j}, Af \rangle$ .

So I have that this limit here is non-negative. I mean, it's a real number. And it's bigger than or equal to  $\epsilon^2/4$ . Now, here's the problem. I have here a sequence of orthonormal vectors, right? And I know that the sum of squares of these Fourier coefficients, which are Fourier coefficients for  $A$  applied to  $f$ , are-- the sum of squares is finite. And therefore, this has to go to 0. And that's the contradiction to the  $\epsilon^2/4$  squared.

So by Bessel, Bessel's inequality, we get that  $\sum_k \langle e_k, Af \rangle^2$ , this is less than or equal to the norm of  $Af$  squared, which is finite. And since this is a convergent series, the individual terms have to converge to 0. And therefore, this equals 0. But this and this are a contradiction.

OK, so that finishes the proof of this theorem about the eigenvalues and eigenspaces for a compact self-adjoint operator. All right, so I think we'll stop there.