[SQUEAKING]

[RUSTLING]

[CLICKING]

VASILY STRELA: So I'll talk mostly about risk-neutral pricing, and then Black-Scholes will be more of an illustration to the method-- to more general method, which I think is very important, and very important for pricing derivatives.

And a few of you did look into Black-Scholes. So my hope is to give a little bit more depth and a little bit more perspective on what's the bigger picture of derivative pricing through risk-neutral valuation.

But as promised, before, let's start from the interest rates from the slide which-- similar slide which I showed you on the first class. And Andrew Gunderson already talked about yield curve construction and everything.

But, yeah, so that's what happened since we started. That's the yield curve-- or it's just a splitting, as it was in September just before beginning of the class, and that's where we are now.

So quite predictable, in a sense, because Fed moved the rates by 75 basis points. And that's exactly what happened here. The parallel shift is only at 70 basis points here, but more of a 50 here.

Now, a little bit of heuristics. How does it look like? Well, right now, it looks like we are very similar to this shape, which was the 19. On the other hand, this shape is almost exactly the same, which was, in fact, just before the crisis.

So, in both cases, a recession-- I mean, in both cases, there, the curves are inverted, and in both cases, a recession followed shortly after. I certainly think-- hope that it is closer to this because just-- this is basically the next snapshot in '21, and the curve is completely normal shape, and the economy was-- the recession was very short and the economy was on a very solid track by then.

And hopefully we are not in this situation because this is just before the crisis. It's before Bear Stearns collapsed, before Lehman collapsed, so a lot of bad things happened after that.

In any case, this is all non-scientific prediction, and the real question here would be to do some thorough data analysis, something like-- Stefan Andreev talked about analyzing the shape of the curves and bond markets using a PCA, Principal Component Analysis.

But always fascinating to look at the data and always fascinating to look at the curves, which, in my opinion, are in the middle-- in the center of finance.

OK, let's move on to the topic of our class. And the idea is to see how replicating the payoff can help us to price derivatives. And how to learn that actually, if we can follow market dynamics, how it can help us to price derivatives.

And actually, that the real-world dynamics is not that important. It's what market implies. That's what important. And how it is connected to martingales and to mathematical apparatus, to stochastic calculus connecting that. But here is a simple example. Horse betting.

So let's assume-- it is clear-- that we have two horses, Horse 1 and Horse 2. And we-- I mean, you're a bookie, and you know the horse as well. You know that this horse has 20% to chance to win and this horse has 80% chance of winning.

But people bet, bet on the market, and 10,000 is bet on this horse. People don't know exactly these probabilities. They know that this horse is better than this. So they bet-- rather, the public bets 50,000 on this and 10,000 on this.

And what you do, you just take a commission. Say 1%. And you have 6,000 in your-- 600 in your pocket of the commission. So now the question is, you need to set the odds for this race.

So, well, one way to set the odds is knowing the real exact probabilities. One, it's reasonable to say, let's make it 4 to 1. So what outcomes are possible? Then horse starts, horses race. So, H1 wins, H2 wins.

So if a horse-- the first horse wins, you will get 50,000, this amount, and you will pay out 4 times this amount because you set the odds 4 to 1. So, this will mean that you will be left with 10,000 profit.

This is low-probability outcome. You know that second horse is better. So what happens if second horse wins? Well, you will keep this money. So it will be 10,000. And you will pay a quarter of this. And this will be, on my books, minus 2,500 at the end. So you will lose, you will have to pay.

Of course, if you compute the expected value, if horses run many, many times, it will be 0. 10 times 20.2 plus-- minus 2, 2 and 1/2 times 0.8 is 0. But for this, on average, the horses have to run a lot. Now, let's consider a different way of setting odds. You forget about these numbers. You just look at what was bad. And you rather set the odds 5 to 1.

OK, let's see what happens. Well, if the first horse wins, you will get 500-- or 50,000, and you'll pay 50,000. Well, if second horse wins, you'll get 10,000, and one-fifth of this, so you pay out 10,000. So no matter which horse wins, you break even for sure in this particular outcome. And if you were collecting some fee, you just keep the fee, you would do it next time.

So, basically what you did by setting the odds according to the market, you hedge yourself. You eliminated every uncertainty, which is probably the outcome which you want.

All right, so this is all about betting, and we are not betting people. In fact, we are scientists. And we'll be talking today about pricing derivatives. Again, some simple examples and derivatives which we will be looking at will be call/put options and forward contracts.

Just to refresh, so what are those? So the simplest possible derivative is a forward contract, which is an agreement to buy a security-- in our case, a stock, in the future for a price which is set today. Generally speaking-- so the payout of a forward is just the difference of the strike and the price at the expiration-- at the expiry. So the payout is S minus K, where K is a strike. Agreed price.

And for forwards, the agreement is that the strike is set in such a way that it costs you nothing to enter it today. You basically say, we agree with you today for a zero price that I'll buy the stock at the price K.

So quite a useful diagram is the payout. That's-- sorry, that's the blue line, so it's-- in our case, it's S minus K. So that's forward. Now, call option. Call option is an option to buy a stock at price K. So basically, at expiry, you will decide, are you buying the stock at price K? Or if you don't want to, you won't.

And generally speaking, if the stock ends at the money, higher than the strike, you will buy it, because it will be cheaper. and if the stock ends below the strike price, you'll say, thank you very much, and the contract will expire worthless.

So this is the payout at the end. And this is the price of the call option today. Obviously, it is not zero-price contract. It is-- you will pay some money to enter into this contract.

Another usual and simple contract would be a put option, which is an agreement to sell the-- it's kind of opposite right. It's basically prevents-- if the option-- the stock is too high, you won't sell it. If the stock ends low, you will sell it for a higher price. So it is basically insurance, insurance against stock falling below the strike price.

And here is the current price. And what we'll be doing today is deriving these curves, deriving the current price, the price today, for the options and the forwards.

And what we'll see is a few observations that the only-- what will matter, it is assumptions on the dynamics of the stock price, some assumptions on how the stock behaves, some assumptions on interest rates. And given that, there will be no uncertainty in the price of the derivative. And this was-- believe me, this wasn't clear to market participants before Black and Scholes.

And the other important observation which we'll see, that the preferences of market participants will not matter. It's in a similarly to our horse-racing example, we'll be able to find the riskless ways to replicate the price of the option.

All right. So, let's start with a simple example of the forward. And let's start from a discrete time and stock prices because it's simpler to understand. And so what does it mean? Is that right now, we have some price of our stock as 0. We have price of our contract, F0. In the case of the forward, we assume that it's 0 because we assume that we are entering at the forward contract at 0 price.

And there are two outcomes that we are only one step away. So some time dt to the future. And at the outcomes, either S1 or S2, so the stock can go either here and here. And our derivative, according to our formula, will go to S1 minus K or S2 minus K.

So, again, from naive perspective, one can say, well, there is some probability for stock to go up to S1 and some probability of 1 minus P for stock to go down. And one would say that we can assume that our price now is actually expected value with this probability of the value of our derivative at the end.

And so it will be P times F1 minus 1 minus P times F2. And-- well, P times S1 minus K plus 1 minus P times S2 minus K. And from here, we can actually-- what we are looking for is the agreed price on our forward. So we can say that our strike price should be PS1 plus 1 minus P times S2.

But we don't know the probability, and in fact, that's, as we'll see in a second, it's not a good way to approach it. And why? Here is why. Because here is a suggestion what we should do. Here is a suggestion how we can construct a risk-neutral portfolio here.

So, that's what we are going to do. So at time 0, we enter into forward contract with strike K, and in a second, we'll see what the strike K will be, at 0 cost. Enter forward at 0 cost.

And we will borrow money to buy the stock. So we're borrow F0. So we'll borrow money in a bank and buy the stock. For a minute, let's assume that our interest rates are 0 for simplicity.

OK. So what happens at expiry? What happens? It's time equals 0, time equal dt. So at expiry, what we'll do, we will deliver the stock because we have it. Deliver S. And we will receive our strike price. And repay the loan.

So basically, what happens here? We borrow here, we had a stock. We came here, we delivered the stock, this costs nothing. So we receive K. That's agreed strike. And we repay the loan, which we started here. Because interest rate is 0, It didn't change. If it changed, it will be some e to the minus-- e to the rt times F0.

So, now, that's what we'll get here. And here, we started with 0 because we paid 0 to enter. What can we say about this? Well, if this was greater than 0, we for sure receive money. There is no other outcome. If it's less than 0, for sure lost money. No arbitrage, assume will tell us that neither is possible.

So because the price was-- initial price was 0, this has to be 0. Because this replicates our payoff. So our strike has to be 0. So if there are no interest rates, the forward contract would just-- we would agree to buy or sell the price-- the stock in the future at the current price if interest rates are 0.

So no choice, and we don't care about real-world probabilities, and we don't care about how the stock behaves in real life. We just replicated everything.

What's interesting, though, what's interesting, though, we can find the probability, which will give us our strike-- or which will give us this formula. So what will it be? Well, well K is equal F0. And so our P-- and I'll call it neutral risk-- neutral probability, S1 plus 1 minus Pn S2. And we can find our Pn from here. S0 minus S2 divided by S1 minus S2.

What's interesting observation here, well, first of all, that we can find this risk-neutral probabilities. And we actually could price our forward contract using these probabilities, using this expected payoff. The interesting-- the most interesting part is that our current stock price our underlying under this risk-neutral probability is expected value of the future move of the stock.

And that's what we will see. So that's basically our risk-neutral measure. And that's how we will be looking for our risk-neutral measure because we can always find a measure which implies that our current underlying value, asset value, is expected value of the future payoff. And that's a measure which we can use to price the derivatives.

OK. So let's move to something more interesting. And let's do a call option, but still in discrete time. So, our call option payoff would be maximum of the value at expiration minus K and 0. That's here. That's our payoff. That's the hockey stick.

But we still assume, for this exercise, let's still assume interest rates are 0. So our diagram will be our value of call option, which we'll be looking for at time 0, our stock is 0-- at S0, stock goes to S1 or to S2.

And our value of our call will go-- we assume that S1 is greater than S0, S2 is less than 0, so here, value of our option will be what? ST minus K. And value of our call option here will be 0, because it will be this thing. OK. Good.

So, now, what we want to do, we want to find the replicating portfolio. And here is suggestion how-- so replicating portfolio in the case of forward was stock itself. It was replicating the payout. Now, my suggestion is to do the following. Our replicating portfolio would be some amount of cash and some amount of stock.

And we want to find these values of cash and how much stock we need such that our option payoff is replicated here. Well, we certainly can do it in this simple case because, well, in the beginning, our portfolio will be B0 plus aS0. At the end, it will be B1 plus S1 or B2 plus aS2.

And, our condition of replication is that in this state, it should be equal to this. And in this state, it should be equal to this because we are replicating our payoff. And let's notice that because our interest rate is 0, again, this thing doesn't change because it's just cash.

Well, nice and easy. From here, we can find that our a-- well, if we subtract this equation from this equation-- I mean, we have two equations with two unknowns. So B0 is one unknown, and the second unknown is a. So from here, we can easily find that a is equal S1 minus K divided by S1 minus S2. And B0 is equal S2 times K minus S1 divided by S1 minus S2.

And, knowing that, if we replicate it the same as before, if we replicated the part-- we have a replication portfolio, that's where it should be here, and this is our C0. And that's what we are looking. We are looking for the value of our option at time 0.

So our C0 is equal to aS0 minus S2. Or it is S1 minus K divided by S1 minus S2 times S0 minus S2. OK. So basically, by the same replication argument, we're able to find our value.

Now, coming back to our risk-neutral, sure enough, if we use this same probability, which is derived from expected value of the stock, the price of our C0 is actually P neutral times ST minus K plus 1 minus P neutral times 0.

So, if we know this probability, and we can derive it from here. Instead of doing all this difficult machinery, we can just take the payoff and take expected value of our payoff using this probability and get the value of our derivative now.

All right. So that's discrete world. And, yeah, so just for your reference, here, it is the same set of formulas for any payout. And one other interesting, useful comment here is that we started with replicating portfolios. So we started with some cash and some stock in order to replicate the value of our option on the next step.

On the other hand, one can rewrite it and say that this portfolio is because this is just cash. This is riskless portfolio. And we'll see just in a second that sometimes it's more convenient, instead of talking about replicating portfolios, it's about talking the portfolios without risk. So we basically form a portfolio of option, of full value of option, and some stock such the combination is riskless.

OK. So that's our discrete world. With that, let's move on to continuous world. So, in continuous world, for today, we'll be looking-- we'll be working with lognormal dynamics, but we'll see that it's not absolutely crucial. It's just a nice example, once again.

And lognormal dynamic is basically dynamic which implies that there is some growth. And there is some random component. Well, one can also write it in this form.

And let's talk a little bit about dW, what it is. So dW is a random walk linear process. So expected value of dW is equal to 0, and standard deviation of dW is equal to square root of dt. And that's what always bothers me. Why standard deviation has to be square root of dt? Why do we know that?

Well, and here is a hand-wavy reason without any matter theory or anything else is the following. So suppose we have dt steps. Our random walk had n steps. So, the collection of those will be dW0 dW1, so on, and dWn.

And the-- not equal. So our dWn, the final value after n steps, it will be sum of all the smaller steps. So, what's expected value of Wn? Well, it is sum of the expected values of all dWs, and each of them is 0, so it is 0. It's, in fact, martingale, it stays 0.

Well, what is the standard deviation, or variance, of our dWn? Well, it will be sum of squared of each step. Yeah, and what I forgot to say-- to tell here, is that probability of us going up-- we are going by 1, either up or down.

And on each step, it is 1/2. With probability 1/2, we go up or down. And correlation, our steps are not correlated. That's important.

So then the variance here will be sum of expected values of dWi squared, each of them will be 1 with probability 1/2. So it will be n. So with probability 1/2, it is 1. With probability 1/2, it's minus 1. So sum of squares will be, well, 1, so n steps, it will be n.

OK, fine. Now, let's put our n steps on a finite period of time. Let's assume that we have a finite period of time and we split it at n equal periods of small delta t. And we also-- let's assume that instead of doing the steps of 1 and minus 1, let's assume that there are some number d. Plus or minus d with probability-- sorry.

OK. So expected value of our random walk still stays 0. And our variance now becomes what? Sum of 1/2 d squared plus 1/2 d squared. So nd squared. And our n, we assume that it is just split period of time. So Td squared divided by delta t right.

So basically, what I'm trying to say is that, if we put our steps on a period of time, that's what's our variance. And now, if we make the steps smaller, the time steps smaller, and the space steps smaller as well, we're kind of taking into infinity. We make-- we are taking a limit over the finite period of time, and the only way for this to stay finite and not 0 is for this to be of order of magnitude of square root of t.

So a long-winded question-- a long-winded answer is, for why standard deviation of our Brownian motion has to be square root of t. Because otherwise, we either will have zero variance or exploding process. So for this process not to explode, this has to be true. All right.

Now, another observation before we go to option pricing is to talk about Taylor formula and Ito formula as the follow-up to it.

So, if our life was deterministic and there was a function of two variables, S and t, then our Taylor formula to the first degree-- or to the second degree, let's write it down, we'll take the partial derivatives with respect of both, and then we'll have second derivatives.

So 1/2 of d 2 f by dt squared dt squared plus 1/2 d 2 f by ds squared. Plus mixed derivative. So this is to second degree plus delta. This is to the second degree.

Now, in our stochastic world, these two terms have different magnitude. So the deterministic term is of magnitude dt, and stochastic term is of magnitude of square root. So, if we go back to our Taylor formula and would like to have expansion to the first degree, what will have to do?

So in our stochastic world-- so this is the first degree. So this we keep. This is first degree, or first degree or less right because our dS is dt plus-- this is of order of magnitude of square root dt. So this we'll keep completely.

And now if we go to mixed terms, this is second degree squarely. Get rid of it. Now, this is certainly more than first degree, so get rid of it. But here, something is left. And what is left is square of this, roughly speaking. So we'll have to have a term which is coming from dS squared, which will be 1/2 of d 2 f by ds squared.

And from ds squared, the first term will be dt squared, not interesting. The mixed term will be dt times dW will be less than dt, but the square of dW will be exactly of the order of magnitude of dt. So here will be sigma squared S squared dt-- dS squared, sorry.

And this is called Ito's formula. Very important for stochastic calculus. And this is quick and dirty derivation of it.

OK. Now, let's move on to our-- now this will leave probably. So we'll need this, this, but this we don't need. So let's move to pricing. So we still want to price some claim, derivative claim. And we'll call it f in our case. And we'll form a replicating portfolio.

And as before, our replicating portfolio will consist of some stock, some cash. And what we want to do is to replicate our value of our claim on each step, on each step dt.

Now, let's do the trick which we did before. Let's rearrange this. So let's notice that aSt minus f of t-- so the combination-- or f minus a-- sorry. That this combination of our derivative and stock is riskless.

And what does it mean? That-- well, now, we'll assume that interest rates are not 0, what does it mean? It means that this thing, if it's riskless, it has to grow with time as e to the rt. By definition. So Bt is equal e to the rt times B0. So we started our strategy at time 0. And since then, our cash is growing with interest rate t.

And what does this mean? Well, this means that our dB is deterministic function. No need for Ito or anything like this. And we all can differentiate exponentials. So trivial mass. That's how rB works.

So, now, that's-- so let's try to differentiate the whole thing. We know the right side, because it's deterministic. It will be this. So, the derivative of our riskless portfolio will be r times our riskless portfolio times dt. Good.

Now, we'll keep it as a right-hand side, and for this, we'll use our-- either Ito's formula. Well, let's do that. So we continue to differentiate that, and we get df minus a dS.

And our f is stochastic, we use our Ito's formula, would be d of dt times dt plus df dS times dS plus our stochastic term, plus 1/2 d 2 f by dS squared dS squared. And we don't forget to subtract this.

And if we just continue further and start collecting terms-- so we'll expand our dS according to our formula. And we'll collect terms with the certain terms with dt and uncertain terms with our dW.

So, if we collecting our terms with dt here, it will be df by dt plus 1/2 coming from dS squared, and dS squared, remember, it will be sigma squared S squared dt. It comes from this only. So it will be-- oh, sorry, sorry, sorry, sorry, sorry. Well, yeah, not-- OK, that's fine. Plus 1/2 d 2 f by dS squared sigma squared S squared.

And I missed this one. And if we are collecting the dt terms, it has to be that. Plus df by dS times mu S. And we should not forget dt term from here minus a mu S, and that completes our dt term.

Now let's see what happens with our dW term. Well, dW term will be partially from here. df by dS sigma S from here. Minus part from here, a sigma S. And this is equal dW-- times dW. And what it is equal to? Well, it is equal to this. Yeah?

STUDENT: How come we can say that dW squared equals dt? Because I thought we just knew that standard deviation is the square root of dt, but, like--

VASILY STRELA: So dW squared is dt.

STUDENT: But I thought W was random. I thought dW was random. So how can you say that when you square it, you get something deterministic?

VASILY STRELA: Well, this is more of a notation, but yeah, we kind of-- yeah, it is not very rigorous notation. But that's basically what Ito's formula tells you. So if you go into the strict proof, that's what you will get, basically, for the second-- for the squared term. But--

PROFESSOR: --add to that, when we talked about Brownian motion, the quadratic variation of Brownian motion is constant over time.

STUDENT: I see.

PROFESSOR: And so over the infinitesimal integrals--

STUDENT: That makes sense, thanks.

VASILY STRELA: A combination of infinitesimal, and the fact that the-- yeah, that the standard deviation is of order of magnitude. But yes, you are right, it is kind of sloppy notation.

OK. And this is equal to rf minus raS dt. Because we knew that this part is deterministic. So there are no dW terms on this side. So what does it mean? That this has to be equal to this, and this has to be equal to 0 because right-hand side here is completely deterministic. So uncertainty has to disappear.

Why it disappears, it's by construction. Because from the very beginning, we said that we want to find such an a, and that's what we'll get just in a second, that this portfolio is riskless. Or another way that we can replicate our derivative.

So, if this is equal to 0, then our a is equal to df by dS. And we are done. So what we are done in a sense, that we arrive at the following equation, basically. Where should I write? How about this?

So our main equation will be this equal to this, but we will substitute a. And let me write it here, the equation which we arrive at. df by dt plus 1/2 d 2 f by dS squared sigma squared S squared is equal to rf minus rdf by dS times S. And this is the Black-Scholes equation. So that's what we are looking for, and that's what we will be solving.

So just to give-- yeah. So in the slides, there are nicer and more cleaner write-up. Yeah. So basically, what we found out, that any derivative claim, any payoff has to satisfy this equation. Well, under assumptions of our lognormal motion, if we assume some other dynamics of the stock, the equation will be different, but we still can derive it in a similar manner.

But the beauty of it that any derivative claim, any payout at any time before expiry has to satisfy this equation. Moreover, this equation doesn't depend on our drift. On our real-world drift. And we will come back to this-- shortly. So it doesn't matter.

What matters is volatility. And we will touch upon-- hopefully we'll have time to touch upon it further in Black-Scholes theory, it is assumed constant, but it doesn't have to be. You can assume some function or you can assume this to be stochastic itself, which will make the equations more complicated to solve, but once again.

And that's what Fisher Black, Myron Scholes, and Robert Merton discovered-- or published in their 1973 paper. Believe it or not, it was quite controversial because-- and actually wasn't very much applauded at the time, but was recognized by a Nobel Prize in '97. And the always fun fact to say that all three of them are associated with MIT. And Bob Merton is still, I think, a professor at MIT, which is obviously quite nice to talk about in these walls.

So now, coming-- well, maybe a couple of remarks. Yeah. Yeah, so which I said that-- so any derivative satisfies this equation, it doesn't depend on real-world drift. We obtained this equation by replicating strategy, and that's the most important.

The other beauty of it is, it's not only replicating strategy, it's also hedging strategy. So if we know our a, which is actually derivative of our current price with respect to stock, we can hedge out all the risk. So basically, by holding option and some amount of stock, this amount of stock, our portfolio becomes riskless, which is very important practical conclusion.

The more mathematical technical observations are that this is actually a form of heat equation, and a good technical exercise would be to figure out what change of variables gives you the heat equation, and this is very well-studied and solved problem, even though it has to be numerical.

So as any partial differential equation, to solve it, it will require the initial-- or actually, in this case, final conditions and boundary conditions. And what is our final condition? Well, our final condition is actually our final payoff, which will be this.

This is our final condition. This we know. And what will be our boundary condition for call? Will be that at 0, the value of the call is 0. And the other boundary condition would be that at infinity, it is actually the-- approaches S minus K.

And if we imply these boundary conditions and final conditions-- and this is the conditions for a put, we can derive-- arrive to analytical or semi-analytical formulas for the prices. Of course, it's expressed in the terms of error function, which has to be tabulated. Yeah.

In more complicated cases, in the cases of more complicated dynamics, of course, the formulas are not so nice, and most probably, it will require some numerical integration and numerical procedures, which is a big part of numerical finance, in fact.

What I wanted to talk about is coming back to risk-neutral pricing. And let me do it here. Basically, how do we find our risk-neutral measure? And how do we find our risk-neutral PDF? Well actually-- we're fine.

In the case of lognormal, it is pretty simple because, well, our dS is equal to mu S dt plus sigma S dW. So, here is a nice trick. If we take expectation value of both sides, then we'll get a differential equation for-- well, for expected value of this is 0 because expected value of dW is equal 0. So an expected value of this will be mu dt. So, nice differential equation for our expected value.

So our-- the expected value of the stock should be equal to initial value of the stock times e to the mu t. But for our risk-neutral assumptions, remember-- so we said that in discrete case, the whole point of risk-neutral probability was that expected value of our stock should be-- our stock may be compounded by interest rate. Well, when interest rate was 0, it was constant. It was-- so in lognormal dynamics, dS over S will be martingale.

So this has to be equal to Se to the rt because under risk-neutral dynamics, our stock has to grow with r. So basically, it tells us that mu has to be equal to r under our risk-neutral measure.

So, now, OK, mu is equal to r, but we can easily find a PDF of this process at time t, and that's what it is, of lognormal process, basically. With mu substituted by r. So, yeah. So mu substituted by r.

And what I'm saying, that now instead of solving our Black-Scholes equation for any claim under lognormal dynamics, we could just take our payoff and integrate it against this PDF, the final PDF, and get the value of the function now as the expected payoff under risk-neutral measure. And that's our risk-neutral measure. So that's a nice benefit.

Of course, it is easier for lognormal, for other processes. It is not that easy, but it's certainly an added benefit.

Another example which I want to give you in the past-- in the last 10 minutes is another application example. So let's form such a portfolio. Let's buy a call and sell a put.

So what is it? Well, let's draw a diagram. And let's draw a diagram at time maturity t. So we long a call so with some strike K. So this is our strike K. So our call looks like this. We short a put. So put would be like this, and short put, it we'll flip it, so it's like this. How does it look? It looks like just like a forward.

So, at maturity, this portfolio has zero uncertainty, it's forward. So at any time before maturity, it has to be the forward value of the stock. So it always has to be. So at maturity, it is St minus K. So-- at maturity. So at any point of time, the difference of call and put has to be the difference of value of the current value of the stock and, to tell the truth, discounted value of the strike. T minus t.

And the beauty of it that we made zero assumptions about dynamics of the stock here. This has to be true always. It's just how the payoffs look like. Because we always know forward, it is only dependent on interest rate. It doesn't depend on dynamics of the stock. And this portfolio doesn't depend on the dynamics of the stock.

And it's quite powerful because it basically tells you that if this is different at any point of time, you have an arbitrage. And to illustrate that, I actually-- when-- sorry. Went to Bloomberg, took real prizes of calls and puts-- this is Apple stock a while ago, admittedly. And here, I literally plotted the data from here for a call and for put and literally summed it up.

And you see that it is a straight line. And no surprise, because if it wouldn't be, it would be an arbitrage, people will jump on it immediately. People check for it all the time.

The beauty of it, that the dynamics of the stock is not lognormal, and how can we see it? Well, I also plotted here implied volatility for-- I think I did it for call here. And if it would be lognormal, the volatility should be straight line, and it is not. That's why we need more complicated models for stock dynamics. That's why Black-Scholes is a nice example, but we need stochastic volatility or others.

Jump diffusion is, by the way, a tricky thing because the martingales with jumps don't really work, so life becomes more complicated. So, yeah. So that's a nice-- another nice example of replication.

And in general, that's how a lot of things function. What you try to do, you look at the payout of a portfolio and you try to replicate the payout with the instruments which you have at hand, or at least approximate the payout, because if you did it, you know the price now. Or you know arbitrage.

The other way to do it is form some complicated portfolio and find misplacement. And then it becomes arbitrage. Nowadays, it's practically impossible, but people did a lot of things with more complicated options even 15, 20 years-- well, 25 years ago.

I think Jake has a good story about that, how he was starting in FX, options. Yeah, just another illustration. I did the same for IBM stock and, yeah, sure enough, it's the same straight line. I didn't plot the implied vol, but it's probably even uglier here.

Yeah, we're about at time, but I would leave you with a little bit of food for thought here. So this is called digital option. The payout of digital option, or all-or-nothing option. It's basically either pays you 0 if you end up out of the money or it pays you 1 if you end up in the money.

This is how Black-Scholes would price it. This is actually how call spread looks from the data which I just showed you. Pretty close to this.

So my challenge to you is find a replicating portfolio which prices a digital because I claim that you don't need to have a dynamic of the stock or do any complicated things here. You can replicate this payout practically exactly, actually approximate the payout. And that's actually how people hedge digital options. Yeah, so think about it.

For with this, let me stop here, and any questions? Happy to answer. Looking forward a little bit, so we have two classes, two invited lectures. And one of them, the one in a week time will be by John Hull. It will be by Zoom, and he will be talking about machine learning.

And I think it will be an interesting lecture-- well, at least interesting because for me, John Hull is an icon. I mean, not only he wrote the most famous derivatives pricing book, he also created Hull-White formulas and made a lot of contribution to quantitative finance.

And we are working on a couple of collaboration projects. He is in the University of Toronto, he is a professor of University of Toronto. So yeah, a little bit of advertisement. All right, thank you very much.