Linear Rates Models

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Overview

- Basic Markets
- Interest Rates
- Rates Products
- Yield Curves
- Hedging
- PnL Attribution
- Etrading
- Bonds

What are linear products?

- Products that have payoffs that are either fixed or linear in rates
- Some of the main types of linear products are
  - Vanilla(ish) Bonds
  - Interest rate swaps
  - Interest Rate Futures
  - Certificates of Deposit
- Models are simpler than options models
- Not so much options modelling to be done here
- So why do we care?

Linear Markets

- The markets for linear products are the largest in existence
  - Bonds have a total of $128T outstanding (per ICMA 2020)
  - IR Swaps outstanding was $573T (per BIS 2023)
  - SOFR Futures open interest > $10T notional
- Very liquid markets
  - US treasures trade $900B daily
  - IRS trade $800B daily
  - SOFR futures trade 4M contracts per day
- Typically, more complex models are built on top of the more vanilla models

Linear Products

- Often used for hedging more complex products
- Usually required as inputs to the complex models
- Previously thought very simple, but post 2008 complexities have raised substantial issues in vanilla modelling
- Some of these issues are
  - Funding
  - Clearing
  - Margining
- Many changes to regulation

Interest Rates

- An interest rate is a rate at which you can earn interest on a principal amount

$$I = N\, r\, \Delta t$$

- The present value of this is the discounted forward value

$$PV = \frac{I}{1 + r_d \Delta t}$$

- Depends on my discounting rate r_d
- Which can be difficult to directly observe
- We will typically assume a SOFR or OIS type rate for discounting

Interest Rates

- Libor was the first commonly used rate
- Invented in the 1960s as a reference rate for syndicated lending
- First IR swaps done in the early 1980s
- Massive growth in the 90s and 00s
  - Total swaps outstanding in excess of $350T
- However, Libor has problems that became more apparent post 2008
  - Libor is an unsecured rate
  - Few transactions like this
- Libor became possible to manipulate
- New reference rates created

SOFR

- The main US reference rate
- Based on the average of overnight government bond repo transactions
  - Approx $800B daily repo trades
- Rate based on the actual transactions observed in the market
- SOFR has some features that complicate matters
  - It's a daily rate
  - Can be compounded into term rates
  - But set in arrears
  - Its also a secured rate – Banks prefer unsecured
- Libor has been phased out over the last few years
- Huge migration (~$350T of swaps on Libor) completed in 2023

SOFR

- SOFR is a daily rate but most products don't pay daily interest, but rather a longer tenor
- SOFR averaged to a term rate
- Compounding

$$r = \frac{360}{d}\left[\prod_{i=1}^{n}\left(1 + \frac{r_i d_i}{360}\right) - 1\right]$$

- Simple Average

$$r = \frac{1}{d}\sum_{i=1}^{n} r_i d_i$$

Other Rates

- Term SOFR
  - Term rate published by the CME based on fitting SOFR futures
  - Used as a reference rate for floating rate coporate loans
  - Is a usual forward looking rate
  - Dealers have restrictions on trading
- FedFunds
  - Another overnight rate based on funding transactions
- Prime, DISCO, MMD, etc, etc, etc

Discounting Rates

- Let assume we have a fixed payment F at time t
- We assume a discounting rate
- The value of this payment today is P
- We purchased this contract by borrowing P at our funding rate
- Now carefully consider all the cashflows and changes in value here
- We have
  - Interest paid on the borrowing
  - Increase in value of the underlying contract

Discounting Rates

- From time t_0 to time t_1 we have interest paid on the funding

$$I(t_1) = P(t_0)\ \frac{r_b}{360}$$

- And the increase in value of the contract

$$P(t_1) = P(t_0)(1 + \frac{r_d}{360})$$

- Or

$$\Delta P = P(t_0)\,\frac{r_d}{360}$$

- These two rates must be the same or we have an arbitrage!
- So the discounting rate is the rate we pay on funding

Interest Rate Products

- There are a number of easily observed interest rate products in the market that activly trade
- We can use the observations of these products to estimate a yield curve
- The most common ones used to build a curve include
  - Certificates of deposit
  - Interest Rate futures
  - Interest Rate swaps

Cash Deposit Rates

- We can invest money for short term at a fixed rate

$$1 = [1 + r(t)\Delta]Z(t)$$

- Or

$$Z(t) = \frac{1}{1 + r(t)\Delta}$$

- These rates are observable and can be used to compute $Z(t)$
- But hard to transact for a swaps desk

IR Futures

- SOFR Futures actively trade on 1M and 3M tenors
  - 3M contract settles on the compounded rate over 3m
  - 3M contract trades on IMM dates
  - 1M contract settled on the average rate over 1M
  - 1M contract trades monthly
- We can use the observed prices as an estimate of the unknown forward rates

$$f(t_0, t_1) = 100 - F - Cvx$$

- There is also a small convexity adjustment Cvx due to the futures margining
  - Can estimate it from a model
  - Or just fit the observed swaps and futures prices

Vanilla Interest Rate Swaps

- An IR swap is an agreement to exchange periodic cashflows over the life of a transaction
  - One side pays a fixed rate that is set on trade date
  - Other side pays a floating rate observed periodically
- Many variations of this structure
- Very useful and flexible instrument
  - Converts fixed rate risk to floating rate risk
  - Off balance sheet
  - Customizable to client requirements
- The most liquid rate derivative product

Swap Flow Rollout

- The logic for rolling out the swap cashflows is superficially simple
  - But has much room for error due to date arithmetic and calendar issues
- Typical USD rollout works as follows
  - Today -> Trade Date
  - Trade Date + Spot Rule -> Effective Date
  - Effective Date + Maturity Tenor -> Unadjusted Maturity Date
  - Unadjusted Maturity Date – N * Payment Frequency -> Unadjusted Roll Dates
  - Unadjusted Roll Dates + Adjustment Rule -> Accrual Dates
  - Start Accrual Date + Reset Date Rule -> Rate Quote Date
  - End Accrual Date + Payment Date Rule -> Payment Date
- Very important to get all the conventions correct

Fixed side valuation

- First we roll out all the cashflows
- Eg a 10Y swap with annual payments will have
  - 10 accrual periods each with a payment date
- Date arithmetic is deceptively simply but the source of many (all?) errors in vanilla swap valuation
- We then value the fixed side as the discounted value of the future payments

$$PV = \sum_{i=1}^{n} C\Delta_i N\, Z(t_i)$$

Floating side valuation

- Similarly we can value the floating side as the discounted expected forward cashflows

$$PV = \sum_{i=1}^{n} f_i \Delta_i N \, Z(t_i)$$

- But here we don't know the floating rate since it sets in the future
  - At the beginning of the period (set in advance), or
  - At the end of the period (set in arrears)
- So we have two rates we need to know to value the swap
  - Discounting rate
  - Index rates
- So what do do here?

Floating Rate Calculation

- We can replicate an unknown forward rate with fixed flows
- Assume we need a rate from t_1 to t_2 and that we can buy zero coupon bonds at t_1 and t_2
- Borrow N at t_0 and invest at rate f from t_1 to t_2
- We assume that this is a fair trade (ie 0 PV at initiation) and find

$$PV = -NZ(t_0) + Nf\Delta Z(t_1) + \mathrm{NZ}(\mathrm{t}_1) = 0$$

$$(1 + f\Delta) = \frac{Z(t_0)}{Z(t_1)}$$

$$f = \frac{1}{\Delta}\left(\frac{Z(t_0)}{Z(t_1)} - 1\right)$$

Implied Swap Rates

- A par swap will have a 0 PV at trade initiation

$$0 = \sum_{i=1}^{n} C\Delta_i N\, Z(t_i) - \sum_{j=1}^{m} f_j \Delta_j N\, Z(t_j)$$

- Or

$$C = \frac{\sum_{j=1}^{m} f_j \Delta_j Z(t_j)}{\sum_{i=1}^{n} \Delta_i Z(t_i)}$$

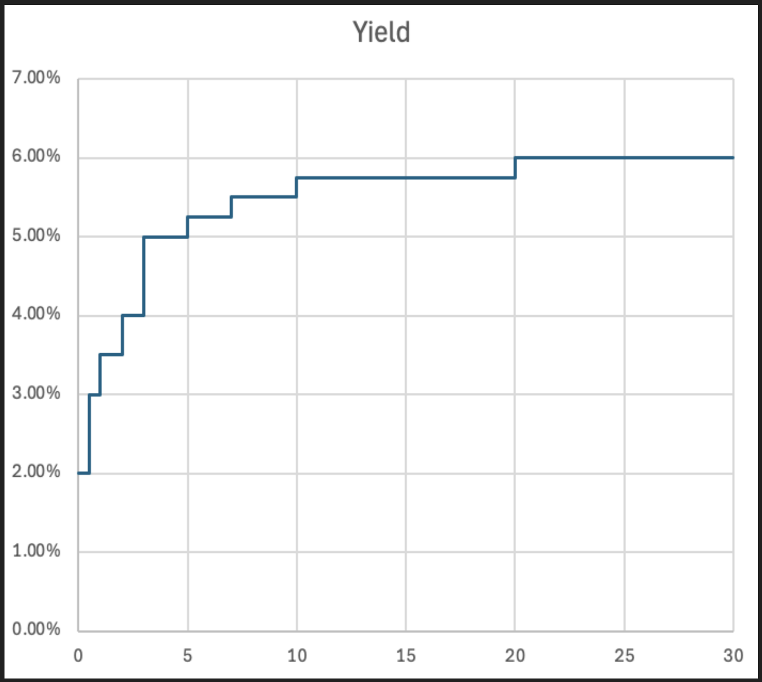
- Par swap rates are observable in the market!

Yield Curve

- First we parameterize our curve by a set of $\left(t_i, Z(t_i)\right)$
- We choose an interpolation method to find $Z(t)$ where $t \notin \{t_i\}$
- There are many possible choices for the knot points $\{t_i\}$
  - Simplest is the set of maturity date for the calibration instruments
  - Can choose other sets such as FOMC meeting dates
  - Choice should be made in the context of your interpolation metho
- We then fit a set of market observables to find $Z(t_i)$

CDF Spline

- A possible interpolation method is to assume that the daily forward rates are constant between knot points
- This is equivalent to
  - Linear interpolation of $t\, r_z$
  - Linear interpolation of $\log(z(t))$
- This is a very fast interpolation method
- And a local interpolation method
- Results in daily forwards with a characteristic shape

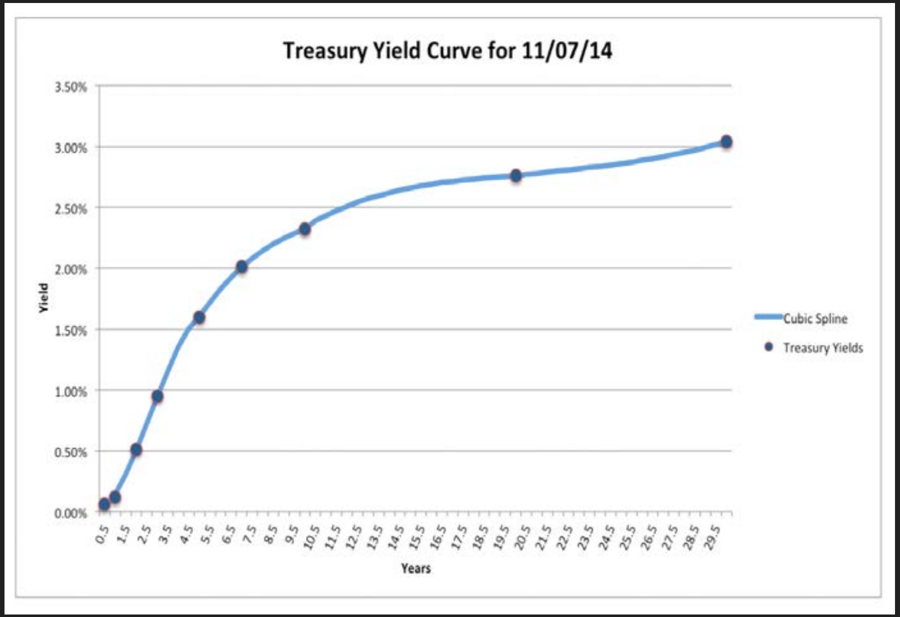


Cubic Spline

- Fit a piecewise cubic polynomial between knot points

$$x_i(t) = \sum_{i=0}^{3} \alpha_{ij}(t - t_i)^i$$

- Impose continuity of x, x', x''
- This is a reasonably fast method that results in a very smooth curve



Curve Calibration

- For each calibration instrument we write the pricing function as a function of the curve

$$E_i = P_i(Z) - P_i(obs)$$

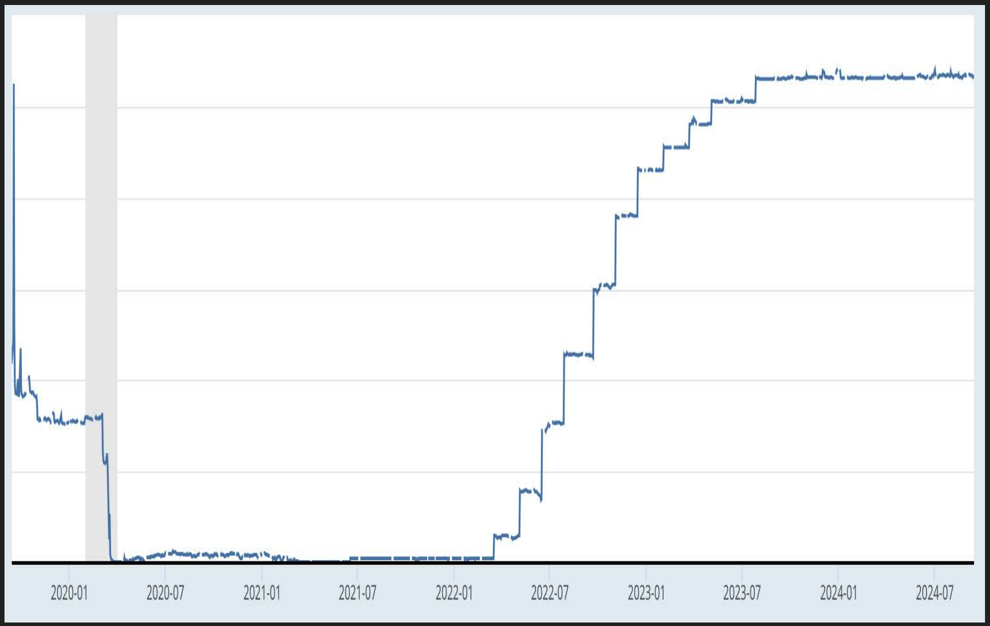
- We then solve to find the curve that results in $E_i = 0$ for all i
- Or alternatively we can do a best fit and minimize the sum of squares objective

$$E = \sum_{i=1}^{N} (E_i(Z))^2$$

- If we have N knot points and N calibration instruments, we can get an exact fit
- We can include other features in the objective function

Historical SOFR Rates

- We can look at historical SOFR rates
- They have an interesting pattern



FOMC Meeting Curve

- We observe that the daily SOFR rate is roughly constant between dates where FOMC changes interest rate policy
- This would seem to be better fit by a CDF interpolation methodology
  - We know the next couple years of FOMC meeting dates
  - Can posit a model where the daily rates are constant between these days
  - Fit observable market instruments
- Need a bit of care in choosing the instruments
  - More meeting dates per year (8) than 3M futures (4)
  - Can include other instruments of constraints
  - FOMC Swaps trade as well for a more direct view on the rates

FOMC Meeting Curve

- However the dates of the FOMC is only known for the next couple years
- And there are not enough instruments past the front end
- So we can create a hybrid curve
  - CDF interpolation at the front of the curve
  - Spline interpolation at the back of the curve
- This works fairly well in practice

Other Curve Approaches

- There are many other features and approaches to building yield curves
- Different interpolators such as tension spline of b-splines
- Can try a best fit approach
- Can fit to within a bid-offer
- Can enforce non-smoothness penalties
- Can build a data driven curve
  - Eg do PCA and build the curve from a small number of factors

Delta Hedging

- We want to compute the exposure of our portfolio to rate moves
- A standard way to do this is to compute the sensitivity of the trades to a small move in the curve inputs
  - Convention is a 1bp move, known as PV01 (or DV01)
- Can use a standard two-sided finite difference

$$\frac{\partial PV}{\partial r} = \frac{PV(r+\epsilon) - PV(r-\epsilon)}{2\epsilon}$$

- Repeat this for each parameter
  - Doing this for each curve input results in a delta ladder (or partial delta)

Gamma Hedging

- We can extend the hedge calculation to include 2nd order effects
  - This is called gamma (or convexity)
- One definition is the difference in Delta for a 1bp move

$$\Gamma = \frac{PV(r+\epsilon) - 2\,PV(r) + PV(r-\epsilon)}{\epsilon}$$

- For partial gamma this is computationally expensive
  - Total PVs requires is N^2
- Other approaches required
  - Can do only parallel gamma
  - Or a factor (PCA) gamma

Hedging

- Some complexities quickly arise
  - Many from the impact of higher order terms (aka convexities)
- How to choose ε ?
  - To big and we can create unusual curve shapes
  - To small and we can be exposed to numerical issues
- Sum of partial derivatives may be different than parallel shift
  - Can do sequential bumps
- You are exposed to your choice of interpolation method
  - Smooth interpolations result in a smeared hedge

PnL Attribution

- One of the key computations for most desks is a PnL attribution
  - Where did you make or lose money?
- Is a good check on how well your model is working
- We know the PnL from t_0 to t_1

$$PnL(t_0 t_1) = PV(t_1) - PV(t_0)$$

- We want to be able to attribute this to our various risk components

PnL Attribution

- We break the PnL into three components
  - Change in portfolio
  - Change due to time
  - Change due to market moves
- Time impact is Theta
  - Much care needed to compute the market data rolls
- The market impact is relatively simple. Eg the linear component is

$$PnL = \sum_{i=1}^{n} \frac{\partial PV}{\partial x_i} \Delta x_i$$

- Portfolio change is theoretically simple but mechanically complex

Clearing

- When you trade with a counterparty, typically the trade is initiated at near 0 PV
- But over time as markets move the PV will move away from 0
- And we now have credit exposure to our counterparty
- Post 2008 clearing became mandatory
- Counterparties assign trades to a CCP
- And post margin to the CCP
- Many clearing houses – LCH, CME, JSCC, etc

CCP Basis

- Dealers required to clear vanilla swaps
  - Both parties assign the swap to a clearing house
  - Reduces credit exposure between dealers
- But a $CME is not exactly the same as a $LCH
- Fixed rates on swaps can be different
- Several reasons for this difference
- Why does this persist?

Tradition

26-Jun-2017

11am (EDT)

Term	Mid Price
1Y	0.10
2Y	0.25
3Y	0.45
4Y	0.70
5Y	1.05
6Y	1.35
7Y	1.55
8Y	1.95
9Y	2.20
10Y	2.45
11Y	2.55
12Y	2.70
15Y	2.85
20Y	3.05
25Y	3.25
30Y	3.40
35Y	3.45
40Y	3.50
50Y	3.55

FI Electronic Trading

- E-trading making inroads into FI
  - About 2/3 of UST volume
- Still well behind Equity, Fx
- Slowly moving up the complexity spectrum
- Many different types of market participants
  - Trading shops
  - Deals
  - Customers

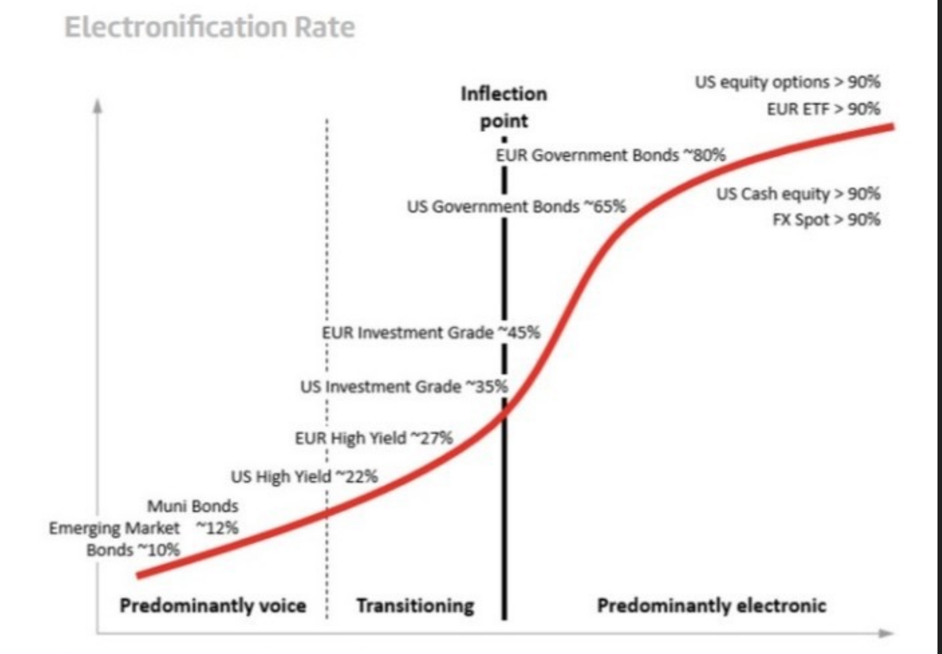


figure 1 *Source: Flow Traders Data 2023

Price Making

- Start with mid price
  - Either from liquidity providers
  - Or from a model
- We then skew the price based on a number of factors
  - Current position
  - Client tier
  - Client toxicity
  - Market volatility
  - Prediction models
- Requires a good integrated system

Fast Model Pricing

- A key feature is speed
- Can be difficult with model-based pricing
- One way to achieve this is linearization

$$r(t) = r(t_0) + \sum_{i=1}^{n} \frac{\partial r}{\partial x_i} \Delta x$$

- Key is to understand the drivers of the price
  - Split into fast components and slower ones
  - Feed the fast ones into the linearization
- Periodically update

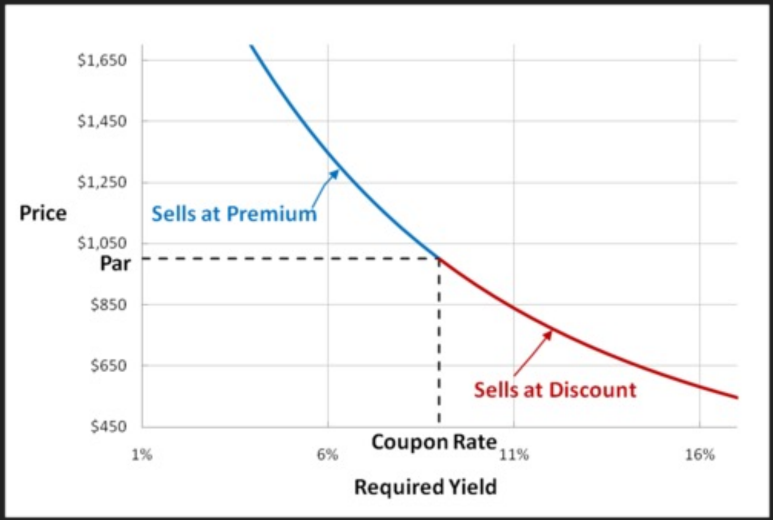
Bonds Again

- The most liquid fixed income instrument in many markets
- Government issue bonds
  - G20 bonds usually consider them to be default afree
  - EM bonds need to consider default
- Despite their apparent simplicity bonds are more complex than swaps when you look very closely
  - Not fungible with each other
  - Many idiosyncrasies

Bond Pricing

- Bonds trade and settle on price
- But yield is a better way to compare different issues
  - Normalizes comparison for different yields and maturities
- Price and yield are related vie the classic p-y equation

$$P_D(y) = \sum_{i=1}^{n} \frac{C_i}{\left(1+\frac{y}{f}\right)^i} + \frac{1}{\left(1+\frac{y}{f}\right)^n}$$

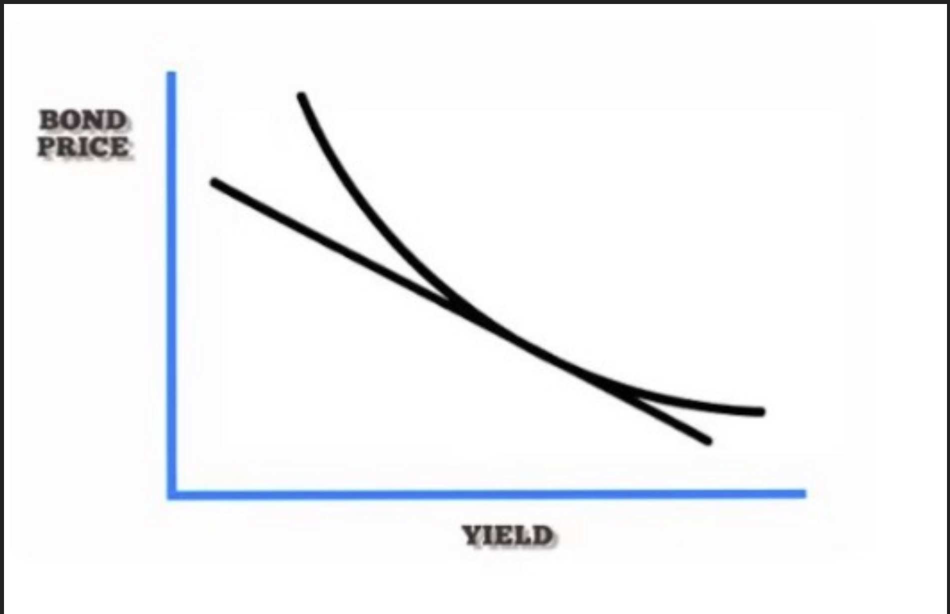


Bond Risk

- A measure of bond risk is the duration
- Mod Duration is the inverse price weighted sensitivity of the bond price wrt yield

$$Mod\ Dur = -\frac{1}{P}\frac{\partial P}{\partial y}$$

- Original was Macaulay duration
  - Cashflow weighted average payment time
  - Is equal to mod duration times (1+y/n)
- Not generic but good to gain insight



Bond Risk

- One issue is how to aggregate bond and swap risk
  - Often use to hedge each other
  - Swaps risk is usually a delta ladder
- We can price a bond off the swaps curve

$$P'_{\mathrm{D}} = \sum_{i=1}^{n} C_i Z(t_i) + Z(t_n)$$

- This will probably not give a price that matches the observed price of the bond
- Define a spread to the discounting rates that results in the correct price

$$P_{\mathrm{D}} = \sum_{i=1}^{n} C_i Z(t_i) e^{-st_i} + Z(t_n) e^{-st_n}$$

- The spread *s* is known as the z-spread

Bond Flavors

- Many different types of bonds
  - Government
  - Asset Backed
  - Corporates
  - Municipals
  - High Yield
  - Many other
- All have nuances in the valuation and risk