Stochastic Processes II

MIT 18.642

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Brownian Motion

History

- 1827, Robert Brown - English Botanist
  zig-zag motion of pollen grains suspended in water
- 1900, Louis Bachelier - French
  price movements in the French bond market.
- 1904, Albert Einstein, diffusion processes
  continuous bombardment of pollen by molecules
  See BrownianMotion2Dim.pdf
- 1923, Norbert Wiener - MIT mathematician
  mathematical foundation for Brownian Motion

Brownian Motion: Definition

Definition Brownian Motion Process

- Continuous-time, continuous state-space Markov Process
- $B(t)$: y component of Brownian Motion in plot verus time t.
- $\{B(t), t \geq 0\}$: complete stochastic process
- $\sigma^2 > 0$: diffusion coefficient
- Normally distributed increments: for every time t and every time increment $\Delta t > 0$,
$$B(t + \Delta t) - B(t) \sim N(0, \sigma^2 \Delta t).$$
- Independence of disjoint increments: for all times $0 \leq t_1 < t_2 \leq t_3 < t_4$,
$$[B(t_2) - B(t_1)] \text{ and } [B(t_4) - B(t_3)]$$
are independent random variables
- $B(0) = 0$ and $B(t)$ is continuous as a function of t

Brownian Motion: Properties

Properties of $\{B(t), t \geq 0\}$

- Markov Process: for any $t > 0$ and any $\Delta t > 0$
  - $B(t + \Delta t) = B(t) + [B(t + \Delta t) - B(t)]$
  - Given information set of $\{B(s), s \leq t\}$, i.e., $\mathcal{F}_t$
    Conditional distribution of $B(t + \Delta t) \mid \mathcal{F}_t$
    equals distribution of $B(t + \Delta t) \mid B(t)$
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$$Cov[B(s), B(t)] \;\; = \;\; E[B(s)B(t)]$$

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$$\begin{aligned} Cov[B(s), B(t)] &= E[B(s)B(t)] \\ &= E[B(s)(B(s) + [B(t) - B(s)])] \\ &= E[B(s)^2] + E[B(s)]E[B(t) - B(s)] \\ &= \sigma^2 s \qquad (+0) \end{aligned}$$

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- Setting $B(0) = x$ for any fixed x maintains all properties
  "Brownian Motion starting at x"

Brownian Motion: Probability Model

Notation for Normal Random Variables

- For a standard normal random variable $Z \sim N(0, 1)$ the pdf and cdf are:

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}, -\infty < z < +\infty$$

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- For a normal random variable $Y \sim N(\mu, \tau^2)$ the pdf and cdf are:

$$p(y) = \frac{1}{\tau}\phi(\frac{y-\mu}{\tau})$$
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- Conditional distribution: $Y = B(t)$ given $X = B(s)$,
$$[Y \mid X = x] \sim N(x, \sigma^2(t-s)) \qquad \text{for } t > s$$

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Brownian Motion: Diffusion Process

Probability Model as a Diffusion

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$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial y^2}$$

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  - See DiffusionNormalDensity.pdf
  - Conservation of probability (energy): for every x
    $$\int_{-\infty}^{+\infty} p(y, t \mid x) dy = 1, \text{ for all } t \geq 0 \ (= s)$$
  - Initial Condition: $x = B(0) = \lim_{t \to 0} B(t)$.
    $$\Longrightarrow \lim_{t \to 0} p(y, t \mid x) = 0, \text{ for } y \neq x.$$
  - $p(y, t \mid x)$ is unique solution
  - Other names: **"Heat Equation"**, **"Fokker-Planck Equation"** **"Kolmogorov Forward Equation"**

Brownian Motion: Limiting Process of Random Walk

Random Walk Process

- $\{X_1, X_2, \ldots\} = \{X_n, n = 1, 2, \ldots\}$: i.i.d. random variables
- $S_n = X_1 + X_2 + \cdots + X_n$ (walk after n i.i.d. steps)
- If $E[X_i] = 0$ and $Var[X_i] = 1$, by **(Central Limit Theorem):**
  $\lim_{n\to\infty} P[\frac{S_n}{\sqrt{n}} \leq c] = \Phi(c)$.

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- Define $\{B_n(t), t \geq 0\}$ by normalizing $\{S_n, n > 0\}$
  $$B_n(t) = \frac{S_{[nt]}}{\sqrt{n}} \text{ , where [nt] = largest integer} \leq nt$$

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See LimitingRandomWalk.pdf

Two Cases: Bernoulli X_i and Gaussian X_i

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Key Properties of Brownian Motion Processes

- Independent increments
- $Var[B(t) - B(s)] \propto |t - s|$.

Brownian Motion: Reflection Principle

Reflecting a Brownian Motion: $\{B^*(t), t \geq 0\}$

- $\{B(t), t \geq 0\}$ Standard Brownian Motion
- Probability space for $\{B(t)\}$ is $\Omega = \{\omega\}$
  (each ω specifies a distinct path $\{(t, B(t)), t \geq 0\}$

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- By symmetry the probability model for each process
  $\{B(t), t \geq 0\}$ and $\{B^*(t), t \geq 0\}$ are *Identical*

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  $\{B(t), t \geq 0\}$ and $\{B^*(t), t \geq 0\}$ are *Identical*
- See ReflectedBrownianMotion.pdf

Brownian Motion: Reflection Principle

Reflecting a Brownian Motion: $\{B^*(t), t \geq 0\}$

- $\{B(t), t \geq 0\}$ Standard Brownian Motion
- Probability space for $\{B(t)\}$ is $\Omega = \{\omega\}$
  (each ω specifies a distinct path $\{(t, B(t)), t \geq 0\}$
- For fixed x, define τ equal to first time $B(\cdot) = x$.
  $\tau = min\{u : B(u) = x\}$.
  (a "Stopping Time" / "Hitting Time")
- Define *Reflected Path* $B^*(u)$
  $$B^*(u) = \begin{cases} B(u) & , \quad \text{for } u \leq \tau \\ x - [B(u) - x] & , \quad \text{for } u > \tau. \end{cases}$$
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Definition: Maximum Variable of $\{B(t), t \geq 0\}$
$M(t) = \max_{0 \leq u \leq t} B(u)$

Proposition: $P[M(t) > x] = 2P[B(t) > x] = 2[1 - \Phi(x/\sqrt{t})]$.

Brownian Motion: Distribution of Maximum

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Proof:

- Fix $x > 0$, and time $t > 0$
- Consider the sets/events of paths

$$A_{END}(t) = \{\omega \in \Omega : B(t) > x\}$$
$$A_{MAX}(t) = \{\omega \in \Omega : M(t) > x\}$$

Claim: $P[A_{MAX}(t)] = 2 \times P[A_{END}(t)]$

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- Suppose path $w_0 \in A_{END}(t)$: $\{(t, B(t \mid \omega_0), t \geq 0\}$
- Set $u = \tau(\omega_0)$ (since $B(t) > x$, $u < t$)

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- By **Reflection Principle** there is a path ω_1:

$B(t \mid \omega_1) = B^*(t \mid \omega_0)$.

By symmetry, w_1 is equally likely as w_0.

So, for every $\omega_0 \in A_{END}$ there is a distinct ω_1

$w_1 \in A_{MAX}$ but $\omega_1 \notin A_{END}$

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$$\begin{aligned} P[A_{MAX}(t)] &= P[A_{MAX}(t) \cap A_{END}(t)] + P[A_{MAX}(t) \cap (A_{END}(t))^c] \\ &= P[A_{END}(t)] + P[A^*_{END}(t)] = 2 \times P(A_{END}) \end{aligned}$$

Brownian Motion: First Hitting Time

Definition: First Hitting Time

For $\{B(t), t \geq 0\}$ with $B(0) = 0$ define

$$\tau = min\{u \geq 0 : B(u) = x\}.$$

Goal: Derive distribution of τ

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- $\tau \leq t \Longrightarrow M(t) \geq x$ and $M(t) \geq x \Longrightarrow \tau \leq t$, so
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- $P[\tau \leq t] = P[M(t) \geq x] = 2P[A_{END}(t)] = 2 \times [1 - \Phi(x/\sqrt{t})]$
- Pdf of τ $f(t \mid x) = \frac{d}{dt}(P[\tau \leq t])$
$$\Longrightarrow f(t \mid x) = \frac{xt^{-3/2}}{\sqrt{2\pi}} e^{-x^2/(2t)},\ 0 < t < \infty$$

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Brownian Motion: Extensions

Reflected Brownian Motion

- Let $\{B(t), t \geq 0\}$ be a Standard Brownian Motion
- Define $\{R(t), t \geq 0\}$:

$$R(t) = |B(t)| = \begin{cases} B(t), & \text{if } B(t) \geq 0 \\ -B(t), & \text{if } B(t) < 0 \end{cases}$$

- Moments of $R(t)$

$$E[R(t)] = \sqrt{2t/\pi} \quad \text{and} \quad Var[R(t)] = (1 - \tfrac{2}{\pi})t$$

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Absorbed Brownian Motion

- Let $\{B(t), t \geq 0\}$ be a Standard Brownian Motion
- Let τ be first hitting time of zero given $B(0) = x > 0$.
- Define $\{A(t), t \geq 0\}$:

$$A(t) = \begin{cases} B(t), & \text{if } t \leq \tau \\ 0, & \text{if } t > \tau \end{cases}$$

- Apply *Reflection Principle* to derive probability model
- Stochastic process for price of an asset which can go bankrupt.

Brownian Bridge

- Let $\{B(t), t \geq 0\}$ be a Standard Brownian Motion
- Define $\{X(t), 0 \leq t \leq 1\}$:

  $$X(t) = B(t) - tB(1).$$

  Note:

  $$\begin{aligned} X(0) &= B(0) - 0B(1) = 0 \\ X(1) &= B(1) - 1B(1) = 0 \end{aligned}$$

- Moments of $B(t)$

  $$E[X(t)] = 0$$

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Normalized Empirical Distribution Function

- $U_1, U_2, \ldots U_n$ i.i.d. $Uniform(0,1)$ and $\hat{F}_n(t) = \frac{\#\{U_i \leq t\}}{n}$
- $\hat{F}_n(t)$: expectation t and variance $(t)(1-t)/n$
- $\sqrt{n}[F_n(t) - t] \longrightarrow Normal(0, t(1-t))$

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Brownian Motion with Drift

- Let $\{B(t), t \geq 0\}$ be a Standard Brownian Motion
- Define $\{X(t); t \geq 0\}$
$$X(t) = \mu t + \sigma B(t), \text{ for } t \geq 0.$$
$\mu =$ **drift parameter**
$\sigma =$ **volatility parameter**

Key Properties of Brownian Motion with Drift

- Independent Increments
- $Var[X(t) - X(s)] \propto |t - s|$
(Same as for Standard Brownian Motion $\mu = 0$, $\sigma = 1$.)

Brownian Motion with Drift

Infinitesimal, One-Step Analysis:

- Conditional Distribution of $X(t+\Delta t)$ given $X(t) = x$
$$\begin{aligned} X(t+\Delta t) &= \mu(t+\Delta t) + \sigma B(t+\Delta t) \\ &= [\mu t + \sigma B(t)] + \mu \Delta t + \sigma[B(t+\Delta t) - B(\Delta t)] \\ &= X(t) + \mu \Delta t + \sigma \Delta B(t) \end{aligned}$$
- Increment of $X(\cdot)$ in terms of increments Δt and $\Delta B(t)$
$$\Delta X = X(t+\Delta t) - X(t) = \mu \Delta t + \sigma \Delta B$$

Properties:

- $E[\Delta X] = \mu \Delta t$
- $Var[\Delta X] = \sigma^2 \Delta t$
- Exact distribution: $\Delta X \sim N(\mu \Delta t, \sigma^2 \Delta t)$.
- As $\Delta t \to 0$
$$\begin{aligned} E[(\Delta X)^2 \mid X(t) = x] &= \sigma^2 \Delta t + (\mu \Delta t)^2 \\ &= \sigma^2 \Delta t + o(\Delta t). \\ E[(\Delta X)^c \mid X(t) = x] &= o(\Delta t) \quad \text{for } c > 2 \end{aligned}$$

Gambler's Ruin Problem

Setup:

- $\{X(t), t \geq 0\}$ Brownian Motion with drift μ and variance σ^2
- Suppose $X(0) = x$ and for levels a and b: $a < x < b$ consider first hitting time

$$\tau = min\{u : X(u) = b \text{ or } X(u) = a\}$$

Problem: Solve for $u(x) = P[X(\tau) = b \mid X(0) = x]$

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Solution: Apply Infinitesimal One-Step Analysis

- Consider Δt so small that $P(\tau \in (0, \Delta t))$ is negligible

$$u(x) = E[u(x + \Delta X)]$$

taking $E[\cdot]$ with respect to r.v. $\Delta X = X(t + \Delta t) - X(t)$.

- Apply Taylor Series to $u(x)$ (assume twice differentiable)

$$\begin{aligned} u(x + \Delta X) &= u(x) + \Delta X u'(x) + \tfrac{1}{2}(\Delta X)^2 u'' + o([\Delta X]^2) \\ \Rightarrow E[u(x + \Delta X)] &= u(x) + E[\Delta X]u'(x) + \tfrac{1}{2}E[(\Delta X)^2]u'' + o(E([\Delta X]^2)) \\ &= u(x) + (\mu\Delta t)u'(x) + \tfrac{1}{2}[\sigma^2\Delta t]u'' + o(\Delta t) \end{aligned}$$

From before:

$$
\begin{aligned}
E[u(x+\Delta X)] &= u(x) + (\mu\Delta t)u'(x) + \tfrac{1}{2}[\sigma^2\Delta t]u'' + o(\Delta t) \\
E[u(x+\Delta X)] &= u(x)
\end{aligned}
$$

$$
\Longrightarrow 0 = (\mu\Delta t)u'(x) + \tfrac{1}{2}[\sigma^2\Delta t]u'' + o(\Delta t)
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General Solution:

$$\begin{aligned} u(x) &= Ae^{-2\mu x/\sigma^2} + B \quad &\text{for} \quad \mu \neq 0 \\ u(x) &= Ax + B \quad &\text{for} \quad \mu = 0 \end{aligned}$$

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For $\mu \neq 0$:

$\Longrightarrow u(x) = \frac{g(x)-g(a)}{g(b)-g(a)}$ with $g(x) = e^{-2\mu x/\sigma^2}$

I.e.

$$P[X(\tau) = b \mid X(0) = x] = \frac{e^{-2\mu x/\sigma^2} - e^{-2\mu a/\sigma^2}}{e^{-2\mu b/\sigma^2} - e^{-2\mu a/\sigma^2}}$$

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See GamblersRuin.pdf

Brownian Motion: Properties of Paths

Proposition With probability 1, the path of a Brownian motion $\{(t, B(t)), t \geq 0\}$ is Not Differentiable at t, for any $t \geq 0$.

Proof:

- Consider the infinitesimal increment of $B(t)$
$$\Delta B(t) = B(t + \Delta t) - B(t) \sim N(0, \Delta t).$$
- Define the normalized increment
$$D(t) = \Delta B(t)/\Delta t \sim N(0, [\Delta t]^{-1}).$$
- If the derivative of $B(t)$ exists then it must satisfy
$$\frac{d}{dt}[B(t)] = \lim_{\Delta t \to 0} D(t).$$
- But for any fixed $M < \infty$,
$$P(|D(t)| \leq M) \to 0, \text{ as } \Delta t \to 0.$$

Note: Order of $\Delta B(t)$ is $O_P(\sqrt{\Delta t}) >> \Delta t$ as $\Delta t \to 0$.

Brownian Motion: Properties of Paths

Theorem (Quadratic Variation) Consider

- $\{B(t), t \geq 0\}$, Standard Brownian Motion
- Partition the interval $[0, T] = \{t : 0 \leq t \leq T\}$ by the set of time points

$$\Pi = \{t_1, t_2, \ldots, t_{n-1}\}$$

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- The Π-**quadratic variation** of $B(\cdot)$ on $[0, T]$ is given by
  $$QV_\Pi([0, T]) = \sum_j [B(t_j) - B(t_{j-1})]^2$$

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Brownian Motion: Properties of Paths

Theorem (Quadratic Variation) Consider

- $\{B(t), t \geq 0\}$, Standard Brownian Motion
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  $$\Pi = \{t_1, t_2, \ldots, t_{n-1}\}$$

- Define $|\Pi| = max(\{|t_{j+1} - t_j|, j = 1, \ldots, n-1\}$

  (with $t_0 = 0$, $t_n = T$)

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Note(!!): $QV([0, T]) = 1$ for virtually every path

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Proof: $\Delta t \equiv T/n$, $QV_\Pi \sim (\Delta t) \times \chi^2(df = n) \longrightarrow T$

See QuadraticVariation.pdf

Adapted Processes

- Stochastic Process $\{X(t), t \geq 0\}$
- **Filtration** of $X(t)$: $\{\mathcal{F}_t, t \geq 0\}$:

  Increasing sigma-fields (set of all events)

  $\mathcal{F}_t \subset \mathcal{F}_{t'}$ for $t' > t$.

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- Suppose $\{Y(t), t \geq 0\}$ is a stochastic process representing a transformation of $X(t)$, where $X(t)$ is the price of a security over time and $Y(t)$ represents the cash flow of a strategy that trades the security.
- If $Y(t)$ is **adapted** to $X(t)$ then
  - $Y(t)$ is a function of only $\{X(u), 0 \leq u \leq t)\}$,
  - $\mathcal{G}_t \subset \mathcal{F}_t$, where

    $$\mathcal{G}_t = \{\text{All Events } \textit{measurable} \text{ with } \{Y(u), 0 \leq u \leq t\}\}$$
  - Transformation of $X(t)$ is not forward-looking.

18.642 Topics in Mathematics with Applications in Finance

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