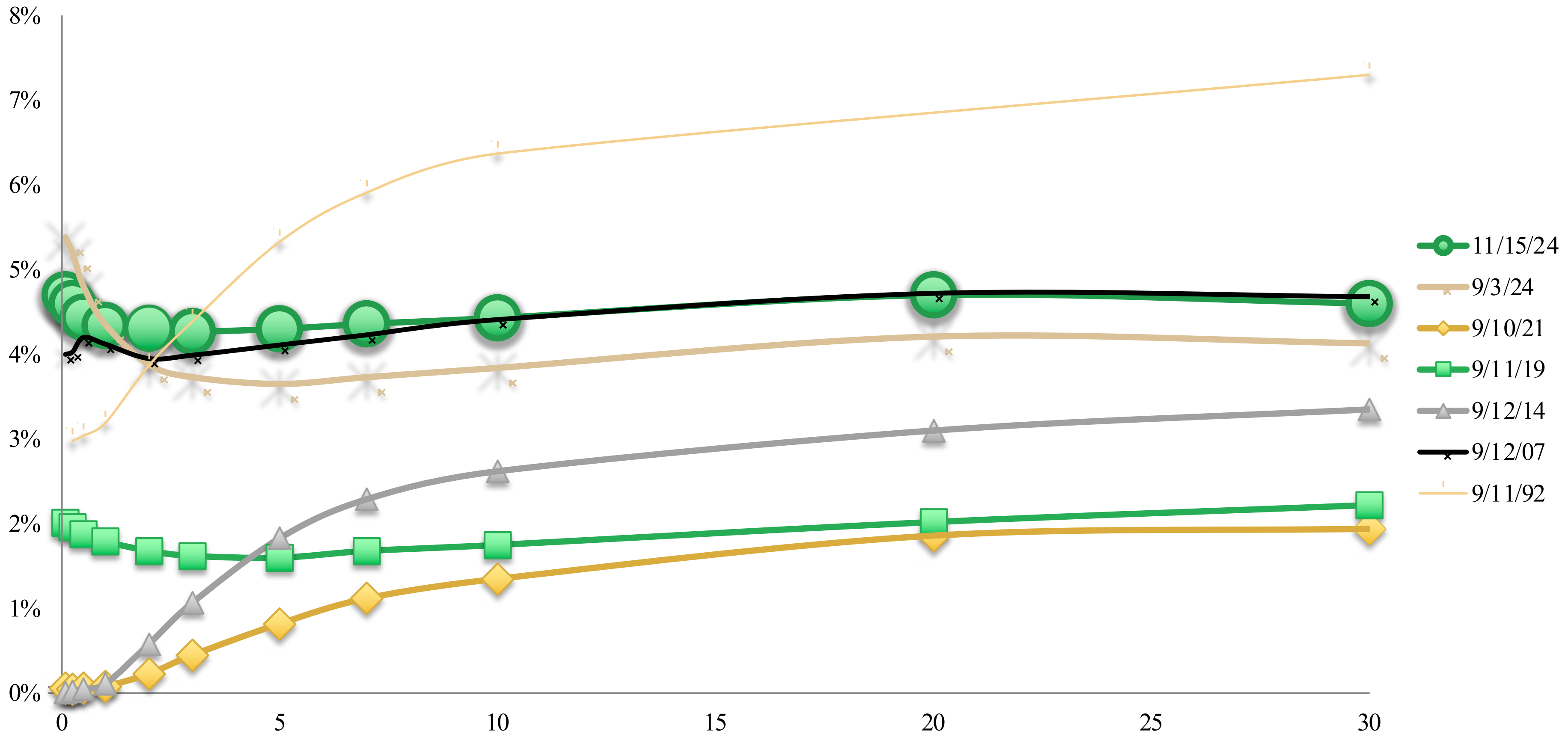
Risk Neutral Valuation
Black-Scholes Equation

Vasily Strela

Yield Curve

- **Term Structure** of yields: Bonds of different maturities may have different yields.
- Discount factors for all times can be bootstrapped from the yield curve



This image is in the public domain.

Source: treasury.gov

Risk Neutral Valuation: Two-Horse Race Example

- One horse has *20%* chance to win another has *80%* chance
- *$10000* is put on the first one and *$50000* on the second

If odds are set 4-1:

- Bookie may gain *$10000* (if first horse wins)
- Bookie may loose *$2500* (if second horse wins)
- Bookie expects to make $0.2 * (10000) + 0.8 * (-2500) = 0$

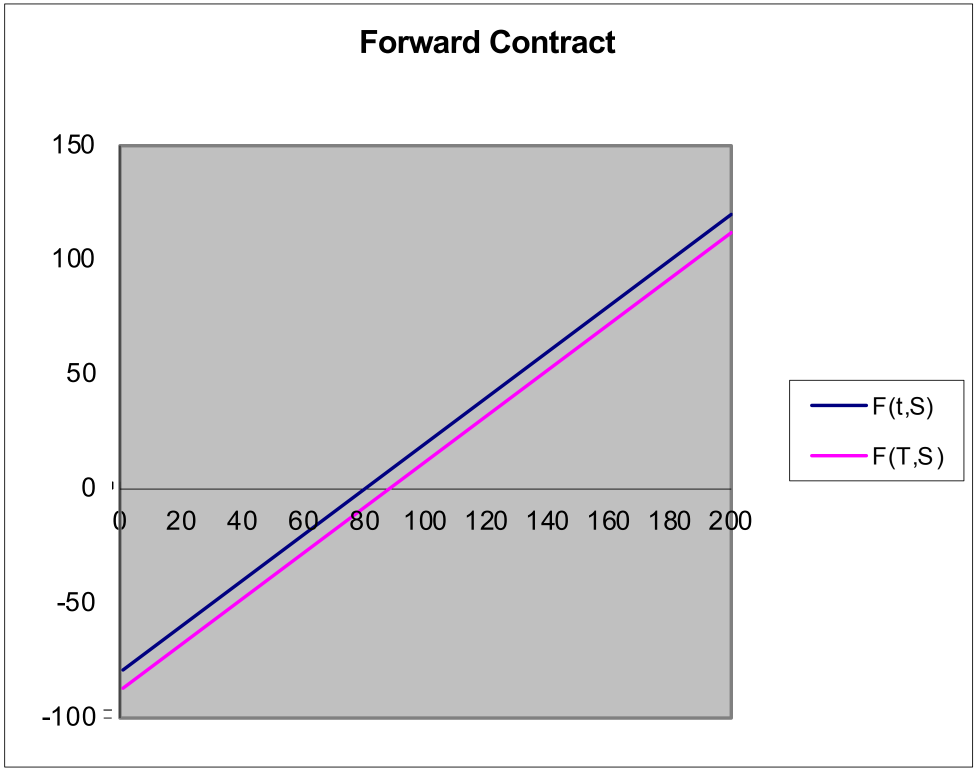
If odds are set 5-1:

- Bookie will not lose or gain money no matter which horse wins

Risk Neutral Valuation : Introduction

We are interested in finding prices of various *derivatives*.

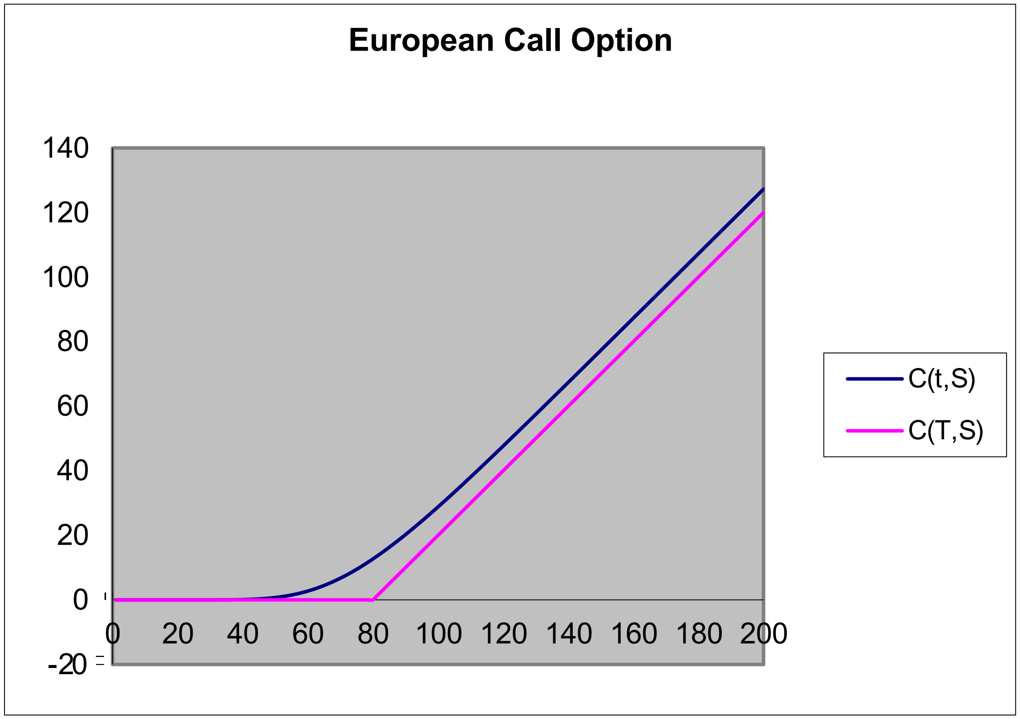
Forward contract pays S-K at time T :



$S(t)=80,\ K=88.41,\ T=2$ (years)

Risk Neutral Valuation: Introduction

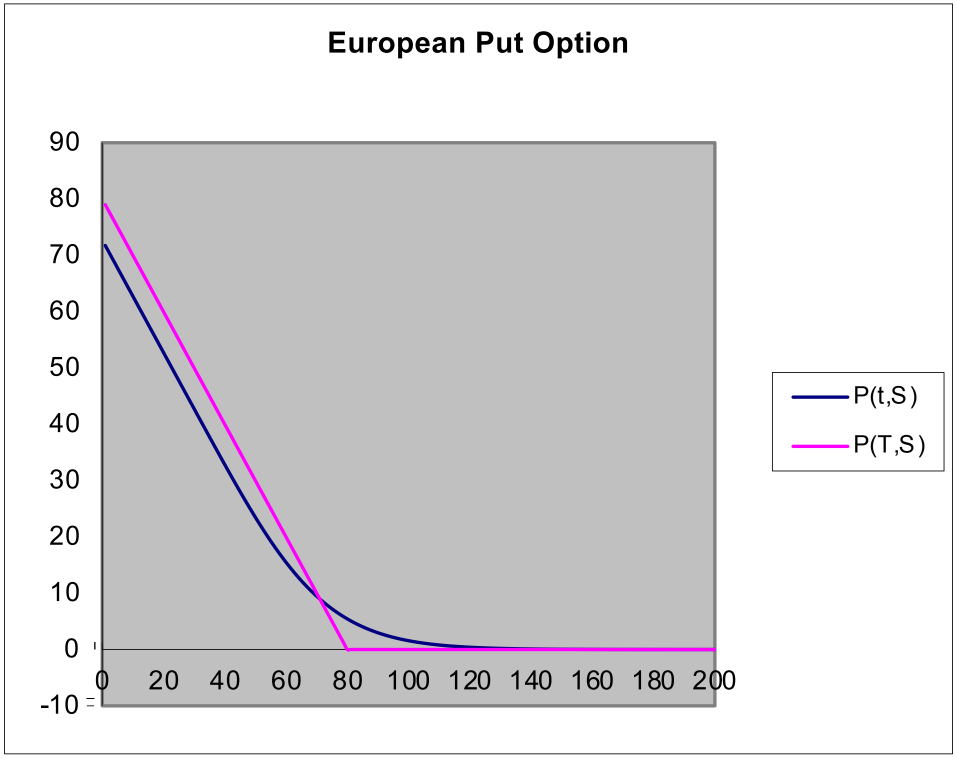
European Call option pays $max(S\text{-}K,0)$ at time T



$S(t)=80,\ K=80,\ T=2$ (years)

Risk Neutral Valuation: Introduction

European Put option pays $max(K\text{-}S,0)$ at time T



$S(t)=80,\ K=80,\ T=2$ (years)

Risk Neutral Valuation: Introduction

- Given current price of the stock and assumptions on the dynamics of stock price, there is no uncertainty about the price of a derivative

- The price is defined only by the price of the stock and not by the risk preferences of the market participants

- Mathematical apparatus allows to compute current price of a derivative and its risks, given certain assumptions about the market

Risk Neutral Valuation: Replicating Portfolio

Consider *Forward* contract which pays S-K in time dt. One could think that its strike K should be defined by the "real world" transition probability p:

$$p(S_1-K)+(1-p)(S_2-K)=pS_1+(1-p)S_2-K$$

$$K_0= pS_1+(1-p)S_2$$

If $p=1/2$, $K_0=(S_1+S_2)/2$

Risk Neutral Valuation: Replicating Portfolio

Consider the following strategy:

1. Borrow $\$S_0$ to buy the stock. Enter *Forward* contract with strike K_0

2. In time dt deliver stock in exchange for K_0 and repay $\$S_0e^{rdt}$

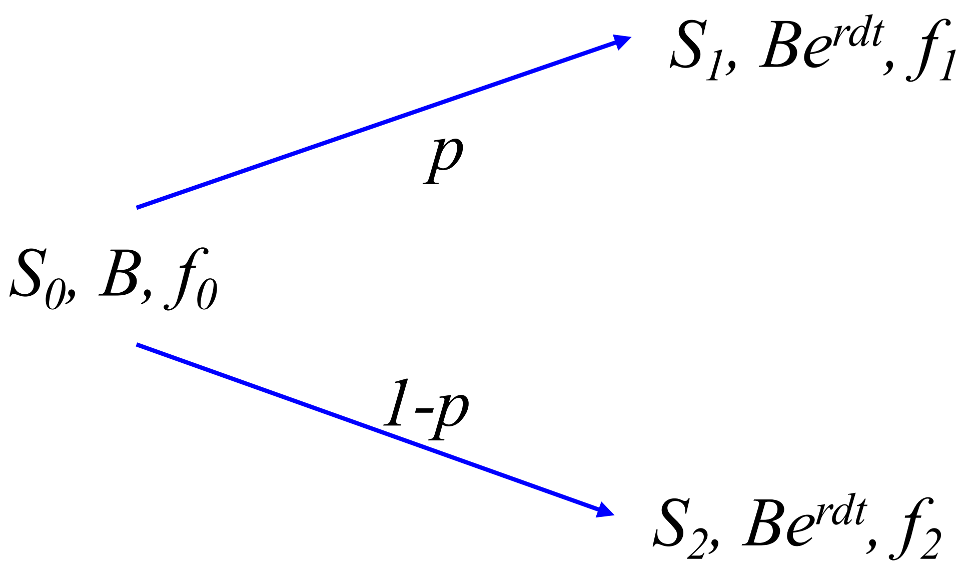
- If $K_0 > S_0e^{rdt}$ we made riskless profit
- If $K_0 < S_0e^{rdt}$ we definitely lost money

$$\Rightarrow K_0 = S_0e^{rdt}$$

Current price of a derivative claim is determined by current price of a portfolio which exactly replicates the payoff of the derivative at the maturity

Risk Neutral Valuation: One step binomial tree

Suppose our economy includes stock S, riskless money market account B with interest rate r and derivative claim f. Assume that only two outcomes are possible in time dt:



Risk Neutral Valuation: One step binomial tree

For a derivative claim f, form a replicating portfolio $f=aS+B$.
Then find a and B such that

$$f_1=aS_1+Be^{rdt}$$
$$f_2=aS_2+Be^{rdt}$$

It implies that

$$f_0=aS_0+B$$

Easy to see that

$$a=\frac{f_1-f_2}{S_1-S_2},\ B=e^{-rdt}\frac{f_2S_1-f_1S_2}{S_1-S_2}$$

$$f_0=e^{-rdt}\left(S_0e^{rdt}\frac{f_1-f_2}{S_1-S_2}+\frac{S_1f_2-S_2f_1}{S_1-S_2}\right)$$

Risk Neutral Valuation: One step binomial tree

One should notice that

$$f_0 = e^{-rdt}\left(f_1 \frac{S_0 e^{rdt} - S_2}{S_1 - S_2} + f_2 \frac{S_1 - S_0 e^{rdt}}{S_1 - S_2} \right)$$

$$f_0 = e^{-rdt}(f_1 p_{rn} + f_2(1 - p_{rn}))$$

where

$$p_{rn}=(S_0 e^{rdt}-S_2)/(S_1-S_2), \quad 0<p_{rn}<1$$

Moreover

$$S_1 p_{rn}+S_2(1-p_{rn})= e^{rdt} S_0$$

Risk Neutral Valuation: Continuous case

$$f_t = e^{-r(T-t)} E_Q(f_T)$$

Q is the risk neutral (martingale) measure under which

$$S_t = e^{-r(T-t)} E_Q(S_T)$$

Black-Scholes equation

Assume that the stock has log-normal dynamics:

$$dS = \mu S dt + \sigma S dW$$

Where dW is normally distributed with mean 0 and standard deviation $\sqrt{dt}$ (i.e. W is a Brownian Motion)

We want to find a replicating portfolio such that

$$df = a dS + dB$$

Black-Scholes equation

Use *Ito's formula:*

$$df(S,t) = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2$$

$$(dS)^2 = \sigma^2 S^2 dt$$

(analogous to first order Taylor expansion, up to dt term)

Black-Scholes equation

$$df = a\,dS + dB$$

Substitute df using Ito's formula, dS, and $dB = rB\,dt$

$$\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu S + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial f}{\partial S}\sigma S\,dW = (a\mu S + rB)dt + a\sigma S\,dW$$

Compare terms:

$$a = \frac{\partial f}{\partial S}, \qquad rB = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2$$

Black-Scholes equation

$B=f\text{-}aS$ is deterministic and as $dB=rBdt$

$$d(f-aS)=r(f-aS)dt$$

Substituting once again $df=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial S}dS+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2dt$ and $a=\frac{\partial f}{\partial S}$

we obtain the **Black-Scholes equation**

$$\frac{\partial f}{\partial t}+\frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2S^2+\frac{\partial f}{\partial S}rS-rf=0$$

Fisher Black, Myron Scholes – paper 1973

Myron Scholes, Robert Merton – Nobel Prize 1997

Black-Scholes equation

- Any tradable derivative satisfies the equation
- There is no dependence on actual drift μ
- We have a hedging strategy (replicating portfolio)
- By a change of variables Black-Scholes equation transforms into heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

Black-Scholes equation

Boundary and *final* conditions are determined by the pay-off of a specific derivative

For European Call

$$C(S,T)=max(S-K,0)$$

$$C(0,t)=0, C(\infty,t)\cong S$$

For European Put

$$P(S,T)=max(K-S,0)$$

$$P(0,t)=Ke^{-r(T-t)}, P(\infty,t)=0$$

Black-Scholes equation

For European Call/Put the equation can be solved analytically

$$C_t = e^{-r(T-t)}\left(e^{r(T-t)}SN(d_1) - KN(d_2)\right)$$

$$P_t = e^{-r(T-t)}\left(KN(-d_2) - e^{r(T-t)}SN(-d_1)\right)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du$$

Black-Scholes: Risk Neutral Valuation

$$f_t = e^{-r(T-t)} E_Q(f_T)$$

Q is the risk neutral measure under which

$$dS = rSdt + \sigma SdW$$

$$pdf(S_T) = \frac{S_t}{\sigma ST\sqrt{2\pi T}} exp\left(-\frac{\left(\ln\left(\frac{S_T}{S_t}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)$$

Black-Scholes equation

For more complicated options or more general assumptions numerical methods have to be used:

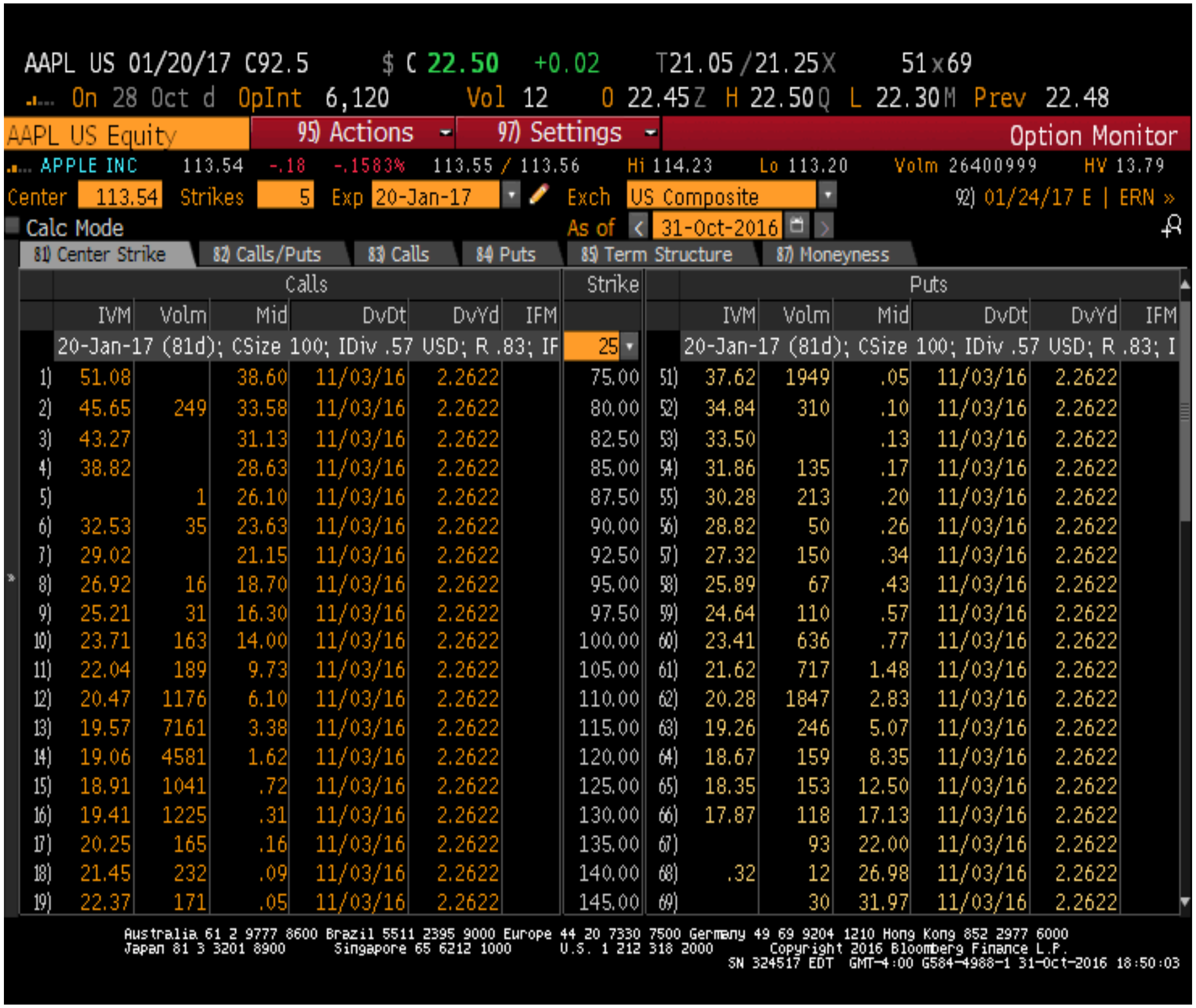
- Finite difference methods
- Tree methods (equivalent to explicit scheme)
- Monte Carlo simulations

Black-Scholes equation: Conclusions

Modern financial services business makes use of

- PDE
- Numerical methods
- Stochastic Calculus
- Simulations
- Statistics
- Much, much more

Risk Neutral Valuation: Example (Call-Put Parity)

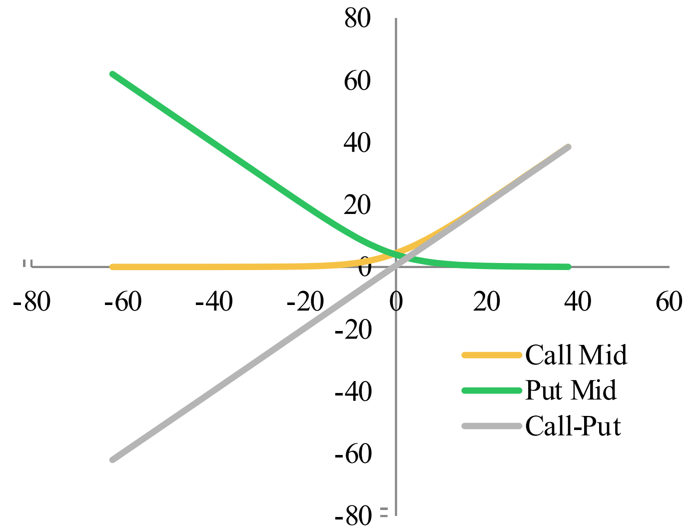


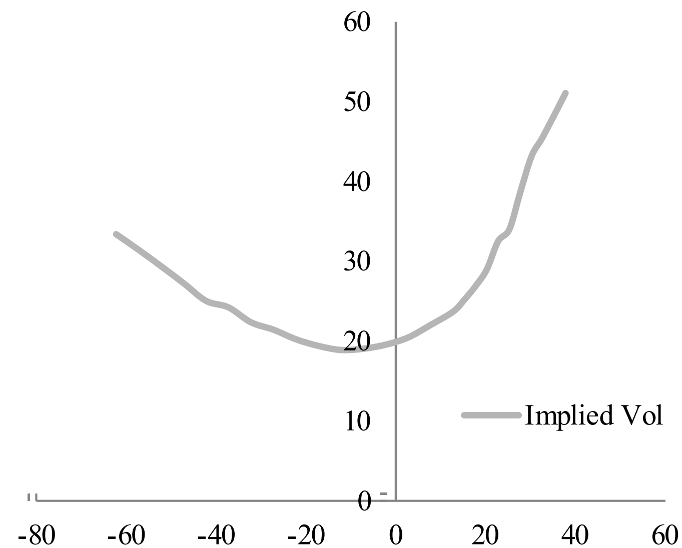
AAPL US 01/20/17 C92.5 $ C 22.50 +0.02 T21.05 / 21.25X 51x69
On 28 Oct d OpInt 6,120 Vol 12 O 22.45Z H 22.50Q L 22.30M Prev 22.48
AAPL US Equity 95) Actions 97) Settings Option Monitor
APPLE INC 113.54 -.18 -.1583% 113.55 / 113.56 Hi 114.23 Lo 113.20 Volm 26400999 HV 13.79
Center 113.54 Strikes 5 Exp 20-Jan-17 Exch US Composite 92) 01/24/17 E | ERN »
Calc Mode As of 31-Oct-2016
81) Center Strike 82) Calls/Puts 83) Calls 84) Puts 85) Term Structure 87) Moneyness

	Calls						Strike		Puts					
	IVM	Volm	Mid	DvDt	DvYd	IFM			IVM	Volm	Mid	DvDt	DvYd	IFM
	20-Jan-17 (81d); CSize 100; IDiv .57 USD; R .83; IF						25		20-Jan-17 (81d); CSize 100; IDiv .57 USD; R .83; I					
1)	51.08		38.60	11/03/16	2.2622		75.00	51)	37.62	1949	.05	11/03/16	2.2622	
2)	45.65	249	33.58	11/03/16	2.2622		80.00	52)	34.84	310	.10	11/03/16	2.2622	
3)	43.27		31.13	11/03/16	2.2622		82.50	53)	33.50		.13	11/03/16	2.2622	
4)	38.82		28.63	11/03/16	2.2622		85.00	54)	31.86	135	.17	11/03/16	2.2622	
5)		1	26.10	11/03/16	2.2622		87.50	55)	30.28	213	.20	11/03/16	2.2622	
6)	32.53	35	23.63	11/03/16	2.2622		90.00	56)	28.82	50	.26	11/03/16	2.2622	
7)	29.02		21.15	11/03/16	2.2622		92.50	57)	27.32	150	.34	11/03/16	2.2622	
8)	26.92	16	18.70	11/03/16	2.2622		95.00	58)	25.89	67	.43	11/03/16	2.2622	
9)	25.21	31	16.30	11/03/16	2.2622		97.50	59)	24.64	110	.57	11/03/16	2.2622	
10)	23.71	163	14.00	11/03/16	2.2622		100.00	60)	23.41	636	.77	11/03/16	2.2622	
11)	22.04	189	9.73	11/03/16	2.2622		105.00	61)	21.62	717	1.48	11/03/16	2.2622	
12)	20.47	1176	6.10	11/03/16	2.2622		110.00	62)	20.28	1847	2.83	11/03/16	2.2622	
13)	19.57	7161	3.38	11/03/16	2.2622		115.00	63)	19.26	246	5.07	11/03/16	2.2622	
14)	19.06	4581	1.62	11/03/16	2.2622		120.00	64)	18.67	159	8.35	11/03/16	2.2622	
15)	18.91	1041	.72	11/03/16	2.2622		125.00	65)	18.35	153	12.50	11/03/16	2.2622	
16)	19.41	1225	.31	11/03/16	2.2622		130.00	66)	17.87	118	17.13	11/03/16	2.2622	
17)	20.25	165	.16	11/03/16	2.2622		135.00	67)		93	22.00	11/03/16	2.2622	
18)	21.45	232	.09	11/03/16	2.2622		140.00	68)	.32	12	26.98	11/03/16	2.2622	
19)	22.37	171	.05	11/03/16	2.2622		145.00	69)		30	31.97	11/03/16	2.2622	

Source: Bloomberg L.P.

$$C(t)-P(t)=S(t)-e^{-r(T-t)}K$$





Risk Neutral Valuation: Example

IBM US $ ↓138.78 +.32 T138.78 / 138.80T 3x3
At 13:54 d Vol 2,077,454 O 137.65N H 139.34D L 137.31D Val 286.859M

IBM US Equity | 95) Actions | 97) Settings | Option Monitor

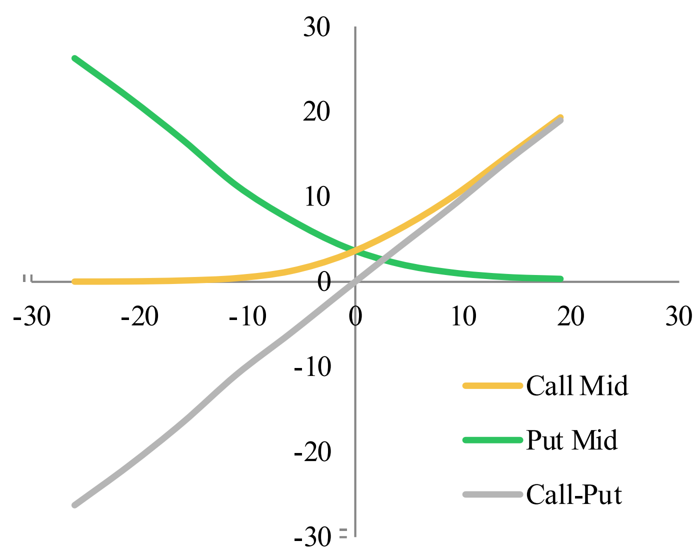
IBM ↓138.78 .32 .2311% 138.78 / 138.80 Hi 139.34 Lo 137.31 Volm 2077454 HV 27.20
Center 138.88 Strikes 5 Exp 18-Dec-15 Exch US Composite 92) 01/19/16 C | ERN»
Calc Mode As of 24-Nov-2015

81) Center Strike | 82) Calls/Puts | 83) Calls | 84) Puts | 85) Term Structure | 87) Moneyness

	Calls						Strike		Puts					
	Ticker	Bid	Ask	Last	IVM	Volm			Ticker	Bid	Ask	Last	IVM	Volm
	18-Dec-15 (24d); CSize 100; R .21; IFwd 138.96						5		18-Dec-15 (24d); CSize 100; R .21; IFwd 138.96					
1)	IBM 12/18/15 C1	3.40	3.55	3.45	17.33	6	137.00	41)	IBM 12/18/15 P1	1.58	1.62	1.54	17.04	146
2)	IBM 12/18/15 C1	2.79	2.87	2.81	16.80	51	138.00	42)	IBM 12/18/15 P1	1.95	1.99	1.86	16.68	51
3)	IBM 12/18/15 C1	2.23	2.26	2.03	16.31	49	139.00	43)	IBM 12/18/15 P1	2.38	2.42	2.30	16.32	26
4)	IBM 12/18/15 C1	1.73	1.79	1.88	16.07	210	140.00	44)	IBM 12/18/15 P1	2.89	2.94	2.70	16.02	188
5)	IBM 12/18/15 C1	1.32	1.37	1.40	15.85	100	141.00	45)	IBM 12/18/15 P1	3.45	3.55	4.30	15.73	1
	15-Jan-16 (52d); CSize 100; R .28; IFwd 139.00						10		15-Jan-16 (52d); CSize 100; R .28; IFwd 139.00					
6)	IBM 1/15/16 C12	17.70	20.95	16.50y	21.10		120.00	46)	IBM 1/15/16 P12	.33	.37	.35	25.60	4
7)	IBM 1/15/16 C12	14.25	15.10	14.47y	22.14		125.00	47)	IBM 1/15/16 P12	.52	.60	.55	22.33	15
8)	IBM 1/15/16 C13	9.30	10.70	10.30	18.67	4	130.00	48)	IBM 1/15/16 P13	1.07	1.10	1.07	20.15	191
9)	IBM 1/15/16 C13	6.05	6.25	6.32	18.96	13	135.00	49)	IBM 1/15/16 P13	2.10	2.17	2.05	18.26	88
10)	IBM 1/15/16 C14	3.05	3.25	3.30	17.52	289	140.00	50)	IBM 1/15/16 P14	4.05	4.20	4.00	16.94	54
11)	IBM 1/15/16 C14	1.23	1.30	1.25	16.44	215	145.00	51)	IBM 1/15/16 P14	7.20	7.40	7.15	15.87	29
12)	IBM 1/15/16 C15	.42	.45	.44	16.12	56	150.00	52)	IBM 1/15/16 P15	10.90	11.75	11.85	16.23	31
13)	IBM 1/15/16 C15	.13	.18	.13	16.49	13	155.00	53)	IBM 1/15/16 P15	15.65	17.75	16.20	24.70	1
14)	IBM 1/15/16 C16	.04	.06	.05	16.97	201	160.00	54)	IBM 1/15/16 P16	20.60	22.75	22.30	26.82	16
15)	IBM 1/15/16 C16	.01	.05	.02	18.84	32	165.00	55)	IBM 1/15/16 P16	24.80	27.75	27.75	24.43	36
	19-Feb-16 (87d); CSize 100; IDiv 1.07 USD; R .39;						5		19-Feb-16 (87d); CSize 100; IDiv 1.07 USD; R .3					
16)	IBM 2/19/16 C13	10.20	11.55	10.88y	19.70		130.00	56)	IBM 2/19/16 P13	2.56	2.61	2.60	21.60	49
17)	IBM 2/19/16 C13	7.35	7.65	6.95	20.94	13	135.00	57)	IBM 2/19/16 P13	4.10	4.20	4.00	20.50	28

Australia 61 2 9777 8600 Brazil 5511 2395 9000 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2015 Bloomberg Finance L.P.
SN 772917 EST GMT-5:00 G714-4634-3 24-Nov-2015 14:09:46

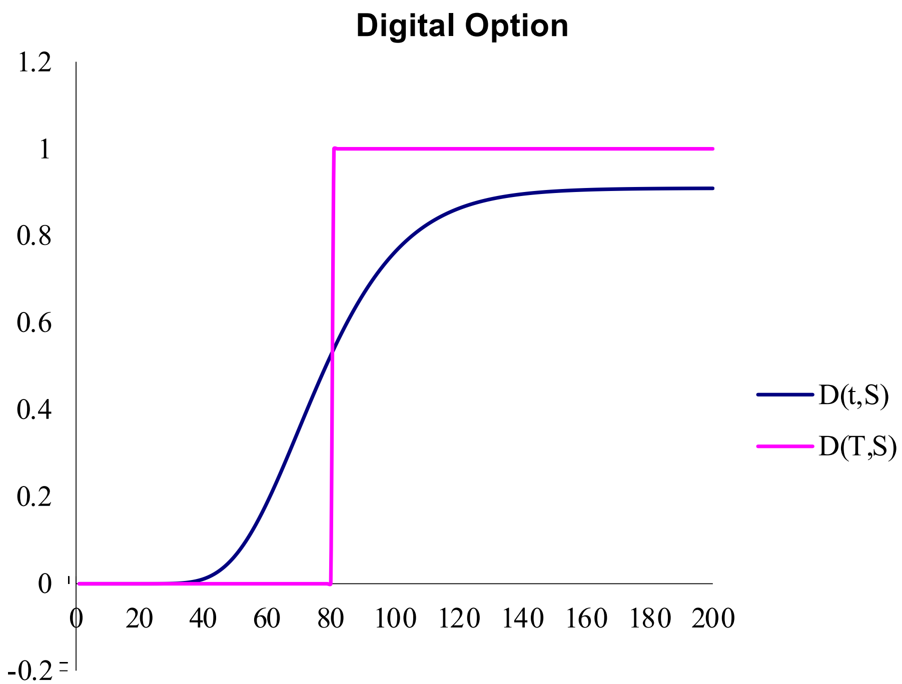
$$C(t)-P(t)=S(t)-e^{-r(T-t)}K$$

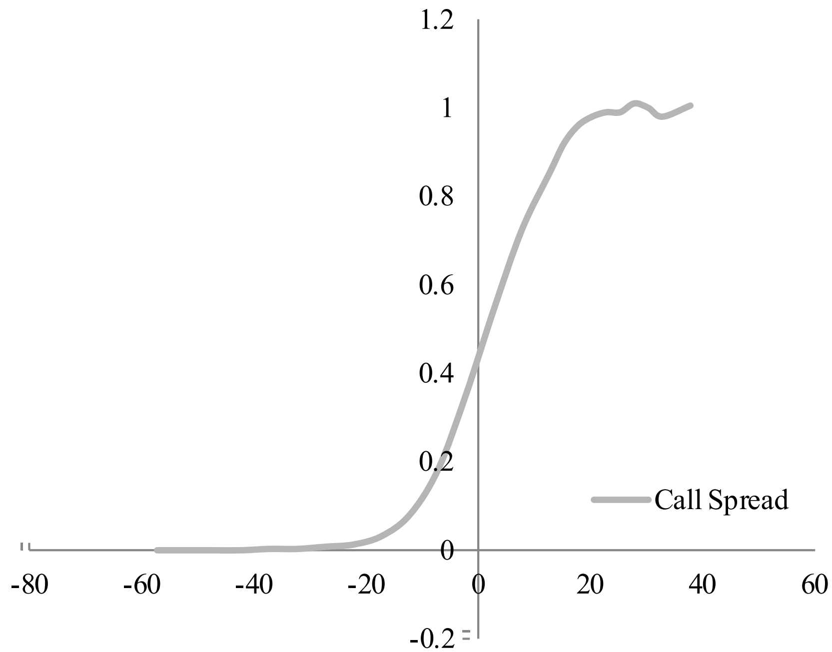


Source: Bloomberg L.P.

Risk Neutral Valuation: Example (Call Spread)

Digital option pays *1* if $S > K$ at time T





$S(t) = K = 80,\ T=2$ (years)

$S(t)=113.54,\ T=81$ (days)