Stochastic Differential Equations

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Black-Scholes Differential Equation

Setup

- $\{P_t, t \geq 0\}$ asset price follows geometric Brownian Motion
  $$dP_t = P_t \mu dt + \sigma P_t dB_t$$
- $\{G_t = G(t, P_t), t \geq 0\}$ price of derivative contingent on P_t.
  (e.g., call option)

By Ito's Lemma

$$dG_t = \left(\frac{\partial G}{\partial P} \mu P_t + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P^2} \sigma^2 P_t^2 \right) dt + \frac{\partial G}{\partial P} \sigma P_t dB_t$$

Discretization of asset price and derivative price

$$\Delta P_t = \mu P_t \Delta t + \sigma P_t \Delta B_t$$
$$\Delta G_t = \left(\frac{\partial G}{\partial P} \mu P_t + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial P^2} \sigma^2 P_t^2 \right) \Delta t + \frac{\partial G}{\partial P} \sigma P_t \Delta B_t$$

Construct portfolio V_t to eliminate the Brownian Motion ΔB_t

Short (sell) one derivative and Buy $\frac{\partial G}{\partial P}$ shares of asset

$$V_t = -G_t + \frac{\partial G}{\partial P}(t, P_t) P_t \quad \text{(portfolio value)}$$

$$\implies \quad \Delta V_t = \left(-\frac{\partial G}{\partial t} - \frac{1}{2} \frac{\partial^2 G}{\partial P^2} \sigma^2 P_t^2 \right) \Delta t \qquad \text{(no } \Delta B_t \text{ component)}$$

Black-Scholes Differential Equation

Portfolio V_t that eliminates Brownian Motion ΔB_t

$$V_t = -G_t + \frac{\partial G}{\partial P}(t, P_t)P_t$$

$$\Delta V_t = \left(-\frac{\partial G}{\partial t} - \frac{1}{2}\frac{\partial^2 G}{\partial P^2}\sigma^2 P_t^2\right)\Delta t \qquad \text{(no } \Delta B_t \text{ component)}$$

- V_t is risk-free and must earn risk-less interest rate r

$$\Delta V_t = V_t \times (r\Delta t)$$

- Equating two expressions for ΔV_t :

$$\begin{aligned} \left(-\frac{\partial G}{\partial t} - \frac{1}{2}\frac{\partial^2 G}{\partial P^2}\sigma^2 P_t^2\right)\Delta t &= V_t \times (r\Delta t) \\ &= (-G_t + \frac{\partial G}{\partial P}P_t)r\Delta t \end{aligned}$$

$\Longrightarrow$ **"The Black-Scholes Differential Equation"**

$$\frac{\partial G}{\partial t} + rP_t\frac{\partial G}{\partial P} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}\sigma^2 P_t^2 = rG_t$$

Black-Scholes Differential Equation

Boundary Conditions Vary by Derivative:

- European call option: $G_T = max(P_T - K, 0)$ where T is time to expiration and K is strike price
- European put option: $G_T = max(0, K - P_T)$.

Heat Equation

Definition: Heat Equation Let $u(x, t) : R \times R^+ \to R$ be real function of space (x) and time (t).

$$\frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2}.$$

Also called **"Diffusion Equation"**

Example 1.1 $u(x, t) =$ temperature in long, thin bar of uniform material; perfectly insulated so temperature only varies with distance x along the bar and with time t.

- Goal: solve initial value problem, given by

  $$u(0, x) = u_0(x), \text{ for } -\infty < x < \infty$$

  for some specific function u_0.

Heat Equation: Properties of Solutions

Observation 1.

- Solutions are linear: if $u_1(x,t)$ and $u_2(x,t)$ satisfy the heat equation, then
  $$u_1(x,t) + u_2(x,t)$$
  also satisfies the heat equation.
- For any collection of solutions $u_s(x,t)$ indexed by $s \in R$,
  $$\int_{-\infty}^{\infty} u_s(x,t) \cdot c(s) ds$$
  is also a solution.
- Strategy: solve general problem with superimposed solutions to easy problems.

Heat Equation: Simple Solution

Observation 2. Simplest initial value problem

- Specify $u(x, 0)$ as a Dirac delta function at $x = 0$

$$u(x,0) = \delta(x):\ \delta(x) = 0, \text{ for } x \neq 0 \text{ and } \int_x \delta(x)dx = 1.$$

- Well-known solution:

$$u_\delta(x,t) = \frac{1}{\sqrt{2\pi \times (2\lambda t)}} e^{-\frac{1}{2}x^2/2\lambda t}$$

(pdf of Normal distribution with mean 0 and variance $2\lambda t$)

- $u_\delta(x,t) \longrightarrow \delta(x)$ as $t \to 0$.
- For fixed $t > 0$ the solution is the probability density function of a normal random variable with mean 0 and variance $2\lambda t$. Simple case: $\lambda = 1/2$ (change time scale or x scale).

Heat Equation: Solution for General Initial Conditions

Observation 3. Consider solving the heat equation for any function $u_0(x) = u(x, 0)$. ($\lambda = 1/2$ case)

- Note that
$$u_0(x) = \int_{-\infty}^{\infty} \delta(x - s) u_0(s) ds$$
- Superimpose the solutions from before:
$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} u_\delta(x - s, t) \cdot u_0(s) ds \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}(x-s)^2/t} \cdot u_0(s) ds. \end{aligned}$$
- If this integration exists (depending on u_0) then
$$\frac{\partial u}{\partial t}(x, t) = \int_{-\infty}^{\infty} \frac{\partial u_\delta}{\partial t}(x - s, t) \cdot u_0(s) ds$$
and
$$\frac{\partial^2 u}{\partial x^2}(x, t) = \int_{-\infty}^{\infty} \frac{\partial^2 u_\delta}{\partial x^2}(x - s, t) \cdot u_0(s) ds$$

$\implies u(x, t)$ satisfies the heat equation too.

Example: Consider $u_0(x)$ as indicator: $1(a < x < b)$

Heat Equation: Gaussian/Normal Structure of Solutions

Observation 3. (continued)

Recap:

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} u_\delta(x-s,t)\cdot u_0(s)ds \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}(x-s)^2/t}\cdot u_0(s)ds. \end{aligned}$$

Consider $u_0(x)$ as indicator: $1(a < x < b)$

- Integrand of $u(x,t)$ is $p(s)\times u_0(s)$ where $p(s)$ is the pdf of a $Normal(x,t)$ distribution.
- $\{X_t, t \geq 0\}$: Brownian motion with zero drift, and variance rate (1) and $X_0 = x$

  $$X_t = x + \int_0^t \sqrt{1} dB_s$$

  $$X_t \mid X_0 = x \sim N(x,t)$$

- $u(x,t) = E_{X_t}[1(a < X_t < b)] = P(X_t \in (a,b) \mid X_0 = x)$

  $= \Phi((b-x)/\sqrt{t}) - \Phi((a-x)/\sqrt{t})$

Solution by Similarity/Invariance

- Consider linear change of time and space variables:
- $\tau = \alpha t$ and $y = \beta x$.
- Define $v(y, \tau) = u(x, t)$.
- Apply the chain rule: $u_t = \alpha v_\tau$, $u_x = \beta v_y$ and $u_{xx} = \beta^2 v_{yy}$.
  PDE: $u_t = \lambda u_{xx} \longrightarrow \alpha v_\tau = \beta^2 \lambda v_{yy}$.
- If $\alpha = \beta^2$, then $v_\tau = \lambda v_{yy}$.
  If $t \to \alpha t$ and $x \to \sqrt{\alpha} x$ then v solves the original problem.
- Strategy: look for solutions that are invariant under
  $$t \to \alpha t \text{ and } x \to \sqrt{\alpha} x$$
  I.e.,
  $$u(x, t) = u(\sqrt{\alpha} x, \alpha t)$$
  Choose $\alpha = \frac{1}{t}$ gives $u(x, t) = u(x/\sqrt{t}, 1) = h(x/\sqrt{t})$.

Heat Equation: Solution by Similarity/Invariance

- Substitute $h(x/\sqrt{t})$ for $u(x,t)$ and the Heat Equation becomes
$$-h'(x/\sqrt{t})\frac{x}{2t^{3/2}} \;=\; \lambda\, h''(x/\sqrt{t})\frac{1}{t}$$
- Substitute $y = x/\sqrt{t}$
$$-h'(y)\tfrac{y}{2} = \lambda h''(y) \quad \text{or} \quad \frac{h''(y)}{h'(y)} = -\frac{y}{2\lambda}$$
- Equivalent expression: $[\log h'(y)]' = -\frac{y}{2\lambda}$
- Integrate for solution to h'
$$h'(y) = ce^{-y^2/4\lambda}$$
- Integrate again for solution to $h(y)$
$$h(y) = \Phi(y/\sqrt{2\lambda}) \quad \text{cdf of } \mathit{Gaussian}(0, 2\lambda)$$
- Substitute $y = x/\sqrt{t}$ and get $u(x,t) = \Phi(x/\sqrt{2\lambda t})$.
- Initial condition for this solution:
$$u(x,0) = \lim_{t\to 0} u(x,t) = \begin{cases} 1 & \mathit{if} \quad x > 0 \\ 1/2 & \mathit{if} \quad x = 0 \\ 0 & \mathit{if} \quad x < 0 \end{cases}$$

Linear Combinations of Solutions

- $u(x,t) = \Phi((x-a)/\sqrt{2\lambda t})$ is a solution for initial condition:
$$u(x,0) = \lim_{t\to 0} u(x,t) = \begin{cases} 1 & \textit{if} \quad x > a \\ 1/2 & \textit{if} \quad x = a \\ 0 & \textit{if} \quad x < a \end{cases}$$
- $u(x,t) = \Phi((x-b)/\sqrt{2\lambda t})$ is a solution for initial condition:
$$u(x,0) = \lim_{t\to 0} u(x,t) = \begin{cases} 1 & \textit{if} \quad x > b \\ 1/2 & \textit{if} \quad x = b \\ 0 & \textit{if} \quad x < b \end{cases}$$
- $u(x,t) = \Phi((x-a)/\sqrt{2\lambda t}) - \Phi((x-b)/\sqrt{2\lambda t})$ is a solution for initial condition:
$$u(x,0) = \lim_{t\to 0} u(x,t) = \begin{cases} 1 & \textit{if} \quad a < x < b \\ 1/2 & \textit{if} \quad x = a \textit{ or } x = b \\ 0 & \textit{if} \quad x \notin [a,b] \end{cases}$$
Note: easy(!) solutions for any step function $u(x,0)$.

Stochastic Differential Equations (SDEs)

SDE: $dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dB(t)$, where

- $\{B(t), t \geq 0\}$ standard Brownian motion
- $\mu(t, x)$ and $\sigma(t, x)$, are functions of space (x) and time (t)

A process $X(t)$ is a *solution* if it satisfies (for all $t \geq 0$)

$$X(t) = \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s).$$

Theorem 2.1 (Existence and uniqueness) If the coefficients of the stochastic differential equation (SDE) with $X(0) = x_0$, and $0 \leq t \leq T$, satisfy the following conditions

- space-variable **Lipschitz condition**
$$|\mu(t, x) - \mu(t, y)|^2 + |\sigma(t, x) - \sigma(t, y)|^2 \leq K|x - y|^2$$
- spatial growth condition
$$|\mu(t, x)|^2 + |\sigma(t, x)|^2 \leq K(1 + |x|^2)$$

then there is a continuous adapted solution $X(t)$ such that

$$\sup_{0 \leq t \leq T} E[X_t^2] < \infty. \quad \text{(uniformly bounded in } L^2(dP))$$

Stochastic Differential equations (SDEs)

Theorem 2.1 (Existence and uniqueness) (continued)
Moreover, if X_t and Y_t are both continuous L^2 bounded solutions of the SDE, then

$$P(X_t = Y_t, \text{ for all } t \in [0, T]) = 1.$$

Conclusion: Many SDEs have solutions which are essentially unique.

Coefficient Matching Method For Solving SDEs:

Consider the SDE for Geometric Brownian Motion

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t), \text{ with } X(0) = x_0 > 0.$$

- Postulate solution: $X(t) = f(t, B(t))$ for some function $f \in C^{1,2}$.
- For such a solution:

$$dX(t) = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \frac{\partial f}{\partial x} dB(t),$$

$$\implies \mu f = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}, \quad \text{and} \quad \sigma f = \frac{\partial f}{\partial x}$$

We can solve successively:

- A soluton to $\sigma f = \frac{\partial f}{\partial x}$ is given by
  $$f(t, x) = e^{\sigma x + g(t)}$$
- Using this in the first equation: $\mu f = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}$
  $$\mu f = g'(t) f + \frac{\sigma^2}{2} f.$$
  Therefore, $g'(t) = \mu - \frac{\sigma^2}{2}$

Which gives
$$f(t, x) = x_0 e^{\sigma x + (\mu - \sigma^2/2)t}.$$
Thus
$$X(t) = f(t, B(t)) = x_0 e^{(\mu - \sigma^2/2)t + \sigma B(t)}.$$

$X(t)$ has a Lognormal(μ_*, σ_*) Distribution

$$\mu_* = (\mu - \sigma^2/2)t \qquad \text{and} \qquad \sigma_* = \sigma t$$
$$E[X(t)] = x_0 e^{(\mu - \frac{\sigma^2}{2})t} \times E[e^{\sigma B(t)}] = x_0 e^{\mu t}$$

Paradox(?) If $0 < \mu < \sigma^2/2$ then:
$$E[X(t)] = x_0 e^{\mu t} \to \infty \text{ while}$$
$$P[X(t) < c] \quad \to 1, \text{ for any fixed } c > 0.$$

Ornstein-Uhlenbeck Processes

Ornstein Uhlenbeck Process: Stochastic differential equation

$$dX_t = -\alpha(X_t - \mu)dt + \sigma dB_t \qquad \text{with } X_0 = x_0.$$

with parameters: $\alpha > 0$, $\sigma > 0$ and $-\infty < \mu < \infty$.

- Drift term: $-\alpha(X_t - \mu)$, negative when $(X_t > \mu)$
  positive when $(X_t < \mu)$.
- Local variability: σ (factor of dB_t) is constant

Applications

- Velocity of gas molecule ($\mu =$ average velocity of molecules)
  Ornstein and Uhlenbeck (1931); mean-reversion toward μ
- Vasicek model of interest rates
  - $\mu =$ target short-term rate of Federal Reserve, or
  - $\mu =$ equilibrium risk-free interest rate of an economy

Modeling Reversion to Mean/Average Level

Ornstein-Uhlenbeck Process: Product-Process Solution

Consider SDE with $\mu = 0$

$$dX_t = -\alpha X_t dt + \sigma dB_t \qquad \text{with } X_0 = x_0.$$

Possible Solution: $X_t = a(t)\left(x_0 + \int_0^t b(s) dB_s\right)$,

where $a(t)$ and $b(s)$ are differentiable functions.

Differential of X_t:

$$\begin{aligned} dX_t &= a'(t)\left(x_0 + \int_0^t b(s) dB_s\right) dt + a(t)b(t)dB_t \\ &= \left[\frac{a'(t)}{a(t)}\right] X_t dt + [a(t)b(t)]\, dB_t \end{aligned}$$

Coefficient Matching:

$$\left[\frac{a'(t)}{a(t)}\right] = -\alpha \quad \textit{and} \quad a(t)b(t) = \sigma$$

- $a(0) = 1 \Longrightarrow a(t) = e^{-\alpha t}$.
- $\Longrightarrow b(t) = [a(t)]^{-1}\sigma = \sigma e^{\alpha t}$.

$$X_t = e^{-\alpha t}\left(x_0 + \int_0^t \sigma e^{\alpha s} dB_s\right) = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

SDE: $dX_t = -\alpha X_t dt + \sigma dB_t$ with $X_0 = x_0$.

Solution: $X_t = x_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$

Properties:

- $E[X_t] = x_0 e^{-\alpha t}$: $\longrightarrow 0$ as $t \to \infty$
- $Var[X_t] = \sigma^2 \int_0^t e^{-2\alpha(t-s)} ds = \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$
  $$\longrightarrow \frac{\sigma^2}{2\alpha} \text{ as } t \to \infty.$$
- As $t \to \infty$ $X_t \to N(0, \sigma^2/2\alpha)$.
- If $x_0 \sim N(0, \sigma^2/2\alpha)$ then so is X_t for all $t > 0$.

Solution for $\mu \neq 0$

$$X_t = \mu + (x_0 - \mu)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

Vector-Representation of Multiple SDEs

$$d\vec{X}_t = \vec{\mu}(t, \vec{X}_t)dt + \sigma(t, \vec{X}_t)d\vec{B}_t, \text{ with } \vec{X}_0 = \vec{x}_0.$$

where

$$\vec{\mu}(t, \vec{X}_t) = \begin{bmatrix} \mu_1(t, \vec{X}_t) \\ \mu_2(t, \vec{X}_t) \\ \vdots \\ \mu_m(t, \vec{X}_t) \end{bmatrix} \qquad d\vec{B}_t = \begin{bmatrix} dB_t^{(1)} \\ dB_t^{(2)} \\ \vdots \\ dB_t^{(m)} \end{bmatrix}$$

$$\sigma(t, \vec{X}_t) = \begin{bmatrix} \sigma_{11}(t, \vec{X}_t) & \sigma_{12}(t, \vec{X}_t) & \cdots & \sigma_{1m}(t, \vec{X}_t) \\ \sigma_{21}(t, \vec{X}_t) & \sigma_{22}(t, \vec{X}_t) & \cdots & \sigma_{2m}(t, \vec{X}_t) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1}(t, \vec{X}_t) & \sigma_{m2}(t, \vec{X}_t) & \cdots & \sigma_{mm}(t, \vec{X}_t) \end{bmatrix}$$

$\{B_t^{(1)}, t \geq 0\}, \ldots, \{B_t^{(m)}, t \geq 0\}$ (independent Br. Motions)

Systems of SDEs

Position and Velocity

- Position: $\{X_t, t \geq 0\}$
- Velocity: $\{V_t, t \geq 0\}$

SDEs With Mean-Reverting Velocity

$$\begin{aligned} dX_t &= V_t dt + \sigma_1 dB_t^{(1)} \\ dV_t &= -\alpha(V_t - \mu)dt + \sigma_2 dB_t^{(2)} \end{aligned}$$

With $\vec{X}_t = \begin{bmatrix} X_t \\ V_t \end{bmatrix}$, $\vec{\mu}(t, \vec{X}_t) = \begin{bmatrix} V_t \\ -\alpha(V_t - \mu) \end{bmatrix}$, and

$$\sigma(t, \vec{X}_t) = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}$$

$$\Longrightarrow d\vec{X}_t = \vec{\mu}(t, \vec{X}_t)dt + \sigma(t, \vec{X}_t)d\vec{B}_t$$

Numerical Methods

- Finite-difference methods
- Monte-Carlo methods
- Tree methods

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