18.642 Problem Set 2 Fall 2024

Collaboration on homework is encouraged, but you will benefit from independent effort to solve the problems before discussing them with other people. **You must write your solution in your own words. List all your collaborators.**

Problem 1. Random Walk with Bias. Let $\{X_n, n = 1, 2, \ldots\}$ be independent and identically distributed steps of a random walk:

- X_n i.i.d. with
$$\begin{aligned} P(X_n = +1) &= p \\ P(X_n = 0) &= 1 - (p + q) \\ P(X_n = -1) &= q \ \ (0 < p), (0 < q), (0 < p + q < 1) \end{aligned}$$
  If $p = q$, then the steps are symmetrically distributed about 0, but if $p > q$, then the steps are biased to be positive.

- $S_n = \sum_1^n X_i, \quad S_0 = 0$

a) Derive the Moment Generating Function of a step $M_{X_i}(t)$. Find the mean and variance of X_i.
b) Derive the Moment Generating Function of the random walk at time n: $M_{S_n}(t)$. Find the mean and variance of S_n

c) Prove that the distribution of $Z_n = \frac{S_n - n\mu}{\sqrt{n\sigma^2}}$ converges to a $Normal(0, 1)$ distribution.
d) Consider modeling the intra-day dynamics of a stock price using a random walk with bias over 78 5-minute steps, i.e., six and one-half hours – the market hours of the NYSE. Suppose the up-step probability $p = 0.52$ and the down-step probability $q = 0.47$, what value of the step size matches an assumption of daily price changes with expectation of +\$0.50?

For this step size, what is the standard deviation of the daily price changes?

Problem 2. Lognormal Distribution. Suppose that X has a $lognormal(\mu, \sigma^2)$ distribution; i.e., $Y = ln(X) \sim N(\mu, \sigma^2)$.

a). Prove that $\mu_* = E[X] = e^{\mu + \frac{1}{2}\sigma^2}$.

b). Prove that $\sigma_*^2 = Var[X] = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = (\mu_*)^2 \left(e^{\sigma^2} - 1\right)$.

Hint: for both a) and b) use the formula for the moment-generating function of a $Normal(\mu, \sigma)$ distribution to compute your answers (you don't need to compute integrals directly).

c). Suppose a stock has price S_0 at $t = 0$ and at time t (years) it has price $S_0 \times X$, where X has lognormal distribution with parameters $\mu = t \times \ln(1.15)$ and $\sigma = \sqrt{t} \times \ln(1.3)$. Find the mean and standard deviation of X (the total return) for horizons $t = 1/12, 3/12, 1$ (years), i.e., corresponding to horizons of 1 month, 3 months and 1 year.

d). Call Option Payoff:

- Option to Buy asset at time $t = T$ (from now,$t = 0$)
  Strike Price: K
- X: price of asset at time T
- Payoff: $C = (X - K)^+ = max(0, X - K)$

If X is a log-normal(μ, σ) random variable and K is a constant, then show that:

$$E[(X - K)^+] = e^{\mu+\sigma^2/2}\Phi(\tfrac{\mu - \ln K}{\sigma} + \sigma) - K\Phi(\tfrac{\mu - \ln K}{\sigma})$$

where $\Phi(c) = \int_{-\infty}^{c} \phi(x)dx$, where $\phi(\cdot)$ is the probability density function of a normal distribution with mean 0 and standard deviation 1.

e) For the stock in part c) if $S_0 = \$100$ compute the expected payoff of the call option when $T = 1/12$ (1 month), $T = 3/12$ (3 months) and $T = 1$ (1 year) for two cases of the strike price:

- $K = \$100$ (an 'At-The-Money' Call)
- $K = \$120$ (an 'Out-of-The-Money' Call)

Comment on the impact of increasing the time to expiration of the call option.

Problem 3. Principal Components (Special Case). Consider a portfolio n assets with fixed weights $w_1, w_2, \ldots, w_n$ For a given horizon, let $X_1, X_2, \ldots, X_n$ be the rate of return on the n assets. The portfolio rate of return is

$$Y = \sum_{i=1}^{n} w_i X_i = \vec{w}^T \vec{X},$$

where $\vec{w}^T = (w_1, \ldots, w_n)$ and $\vec{X}^T = (X_1, \ldots, X_n)$.
Define $\vec{\mu} = (\mu_1, \ldots, \mu_n)^T$, the vector of expectations of the n assets and $\Sigma = ||\sigma_{i,j}||$, the $n \times n$ covariance matrix of the n assets, i.e.,

$$\Sigma_{i,j} = \sigma_{i,j} = Cov(X_i, X_j)$$

a) Show that $E[Y] = \vec{w}^T \vec{\mu}$ and $Var[Y] = \vec{w}^T \Sigma \vec{w}$

b) Prove that Σ is a positive semi-definite matrix (for *any* set of random variables $X_1, \ldots, X_n$).

c) Consider the special case where all elements of the covariance matrix Σ are positive: $\Sigma_{i,j} > 0$, $1 \leq i, j, \leq n$.

Prove that the largest eigenvalue of Σ, λ_{MAX} has multiplicity 1, with eigenvector $\vec{v}$ which can be defined such that each component is positive: $v_i > 0$, $1 \leq i \leq n$.

d) Let $PC1$ be the first principal component variable of $\vec{X} = [X_1, \ldots, X_n]^T$. Write a formula for $PC1$ in terms of $\vec{X}$, μ, and $\vec{v}$. What is the expectation (mean) and variance of $PC1$?

Problem 4. Possible Group Project Topics

In coming weeks the class will be divided into groups of 3-4 students who will undertake a group project on some quantitative finance topic. The group will prepare an in-class presentation and a written lecture note for distribution to classmates. Prepare separate descriptions of 2-3 topics you would like to pursue for the group project. For each topic include:

- Topic Title
- Description (one or two paragraphs with detailed description, including possible project objectives)

These topic descriptions will be compiled and distributed to the class to help the formation of groups.

18.642 Topics in Mathematics with Applications in Finance
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