18.642 Assignment: Problem Set 5 Fall 2024

Collaboration on homework is encouraged, but you will benefit from independent effort to solve the problems before discussing them with other people. **You must write your solution in your own words. List all your collaborators.** Levy Processes extend/generalize Brownian Motion, continuous-time models of random walks. A formal definition is now given.

Definition: A **Levy Process**, $\{L(t), -\infty < t < +\infty\}$ is a process with the following properties:

- $L(0) = 0$
- $L(t) - L(s)$ has the same distribution as $L(t - s)$ for all s and t such that $s \leq t$.
- If (s, t) and (u, v) are disjoint intervals, then
  $L(t) - L(s)$ and $L(v) - L(u)$ are independent.
- $\{L(t)\}$ is continuous in probability, i.e., for all $\epsilon > 0$, and for all $t \in R$
  $\lim_{s \to t} P(|L(t) - L(s)| > \epsilon) = 0$.

1. Suppose $\{W(t)\}$ is a Brownian motion model with drift $\mu \in R$ and volatility $\sigma > 0$.

   1(a) Prove that $\{W(t)\}$ is a Levy Process.

   1(b) Simulate the path of a Brownian Motion Process with the following properties:
   - $T = 1$(year)
   - $\mu = 0.30$, $\sigma = 0.40$
   - $m = 252$ increments, i.e., $t_i = i \times (T/m)$, $i = 1, 2, \ldots, m$
   - Plot five simulated paths of the process (plotting just the process values at the time increments).

2. The stochastic process $\{N(t), t \geq 0\}$, is a Poisson process with intensity or jump-rate λ if:

   - For any n times: $0 = t_0 < t_1 < \cdots < t_n$, the increments $N(t_k) - N(t_{k-1})$ are independent, for $1 \leq k \leq n$.
   - The distribution of $N(t) - N(s)$ is Poisson with (mean) parameter $\lambda(t - s)$, for all $0 \leq s < t < \infty$.

2(a) Prove that $\{N(t)\}$ is a Levy Process.

2(b) Simulate the path of a Poisson Process with the following properties:

- $T = 1$(year)
- $\lambda = 24$ (rate of 24 events per year)
- $m = 252$ increments, i.e., $t_i = i \times (T/m)$, $i = 1, 2, \ldots, m$
- Plot five simulated paths of the process (plotting just the process values at the time increments).

3. Extreme Value Method for Estimating the Variance of the Rate of Return.

3(a) Read the article Parkinson (1980).

3(b) Explain the definition of the term D_x in Section II.

3(c) Explain the definition of the term D_l in Section III.

3(d) Using the results of Section IV, explain why the improvement using D_l, as an estimate of V (in Section V) could be of particular importance in studies of the time and price dependence (if any) of V.

4. Volatility Case Studies

The following R Markdown files produce the volatility case study pdfs for the SP500 and for the stock ARKK.

- $Case\_Study\_Estimating\_SP500\_Historical\_Volatility.Rmd$
- $Case\_Study\_Estimating\_ARKK\_Historical\_Volatility.Rmd$

The case study analyses use time series forecasting methods detailed in the text:

Hyndman, Rob J, and Athanasopoulos, George (2018) Forecasting: Principles and Practice (2nd ed), `https://otexts.com/fpp2/`

Chapter 3 covers the naive method and residual diagnostics. Chapter 8 covers the automatic specification of Arima and seasonal Arima models. The package $fpp2$ includes all functions used in the text.

4(a) Detail/comment on specific results of the SP500 case study you consider interesting/significant.

4(b) Detail/comment on specific results of the ARKK case study you consider interesting/significant.

4(c) Compare the results/findings about volatility for ARKK versus the SP500.

5. (Optional/Extra Credit) Choose your own stock symbol and revise the volatility case study markdown file of ARKK to conduct the same case study analysis of your stock.

5(a) Create the pdf file of the case study.

5(b) Detail/comment on specific results you consider interesting/significant.

5(c) Compare the results/findings for your stock to those for the SP500 and ARKK.

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