

15 Elliptic curves over \mathbb{C} (part 2)

Last time we showed that every lattice $L \subseteq \mathbb{C}$ gives rise to an elliptic curve

$$E_L: y^2 = 4x^3 - g_2(L)x - g_3(L),$$

where

$$g_2(L) := 60G_4(L) := 60 \sum_{L^*} \frac{1}{\omega^4}, \quad g_3(L) := 140G_6(L) := 140 \sum_{L^*} \frac{1}{\omega^6},$$

with $L^* := L - \{0\}$, and we defined a map

$$\Phi: \mathbb{C}/L \rightarrow E_L(\mathbb{C})$$

$$z \mapsto \begin{cases} (\wp(z), \wp'(z)) & z \notin L \\ 0 & z \in L \end{cases}$$

where

$$\wp(z) = \wp(z; L) = \frac{1}{z^2} + \sum_{\omega \in L^*} \left(\frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right)$$

is the Weierstrass \wp -function for the lattice L , and

$$\wp'(z) = -2 \sum_{\omega \in L} \frac{1}{(z - \omega)^3}.$$

In this lecture we will prove two theorems. First we will prove that Φ is an isomorphism of additive groups; it is also an isomorphism of complex manifolds [3, Cor. 5.1.1], and of complex Lie groups, but we won't prove this right now.¹ Second, we will prove that every elliptic curve E/\mathbb{C} is isomorphic to E_L for some lattice L ; this is the *Uniformization Theorem*.

15.1 The isomorphism from a torus to the corresponding elliptic curve

Theorem 15.1. *Let $L \subseteq \mathbb{C}$ be a lattice and let $E_L: y^2 = 4x^3 - g_2(L)x - g_3(L)$ be the corresponding elliptic curve. The map $\Phi: \mathbb{C}/L \rightarrow E_L(\mathbb{C})$ is a group isomorphism.*

Proof. We first note that $\Phi(0) = 0$, so Φ preserves the identity, and for all $z \notin L$ we have

$$\Phi(-z) = (\wp(-z), \wp'(-z)) = (\wp(z), -\wp'(z)) = -\Phi(z),$$

since \wp is even and \wp' is odd, so Φ is compatible with taking inverses.

Let $L = [\omega_1, \omega_2]$. There are three points of order 2 in \mathbb{C}/L : $\omega_1/2, \omega_2/2$, and $(\omega_1 + \omega_2)/2$. By Lemma 14.31, \wp' vanishes at these points, hence Φ maps points of order 2 in \mathbb{C}/L to points of order 2 in $E_L(\mathbb{C})$, since the latter are the points with y -coordinate zero. Moreover, Φ is injective on points of order 2, since $\wp(z)$ maps each point of order 2 in \mathbb{C}/L to a distinct root of $4\wp(z)^3 - g_2(L)\wp(z) - g_3(L)$, as shown in the proof of Lemma 14.33. The restriction of Φ to $(\mathbb{C}/L)[2]$ defines a bijection of $(\mathbb{C}/L)[2] \xrightarrow{\sim} E_L[2] \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with $\Phi(0) = 0$, which must be a group isomorphism.

¹This is not difficult to show, but it would distract us from our immediate goal. We will see an explicit isomorphism of complex manifolds in a few lectures when we study modular curves, and in that case we will take the time to define precisely what this means and to prove it.

To show that Φ is surjective, let $(x_0, y_0) \in E_L(\mathbb{C})$. The elliptic function $f(z) = \wp(z) - x_0$ has order 2, hence it has two zeros in the fundamental parallelogram \mathcal{F}_0 , by Theorem 14.18. Neither of these zeros occurs at $z = 0$, since f has a pole at 0. So let $z_0 \neq 0$ be a zero of $f(z)$ in \mathcal{F}_0 . Then $\wp(z_0) = x_0$, which implies $\Phi(z_0) = (x_0, \pm y_0)$ and therefore $(x_0, y_0) = \Phi(\pm z_0)$; thus Φ is surjective.

We now show that Φ is injective. Let $z_1, z_2 \in \mathcal{F}_0$ and suppose that $\Phi(z_1) = \Phi(z_2)$. If $2z_1 \in L$ then z_1 is a 2-torsion element and we have already shown that Φ restricts to a bijection on $(\mathbb{C}/L)[2]$, so we must have $z_1 = z_2$. We now assume $2z_1 \notin L$, which implies $\wp'(z_1) \neq 0$. As argued above, the roots of $f(z) = \wp(z) - \wp(z_1)$ in \mathcal{F}_0 are $\pm z_1$, thus $z_2 \equiv \pm z_1 \pmod{L}$. We also have $\wp'(z_1) = \wp'(z_2)$, and this forces $z_2 \equiv z_1 \pmod{L}$, since $\wp'(-z_1) = -\wp'(z_1) \neq \wp'(z_1)$ because $\wp'(z_1) \neq 0$.

It remains only to show that $\Phi(z_1 + z_2) = \Phi(z_1) + \Phi(z_2)$. So let $z_1, z_2 \in \mathcal{F}_0$; we may assume that $z_1, z_2, z_1 + z_2 \notin L$ since the case where either z_1 or z_2 lies in L is immediate, and if $z_1 + z_2 \in L$ then z_1 and z_2 are inverses modulo L , a case treated above.

The points $P_1 = \Phi(z_1)$ and $P_2 = \Phi(z_2)$ are affine points in $E_L(\mathbb{C})$, and the line ℓ between them cannot be vertical because P_1 and P_2 are not inverses (since z_1 and z_2 are not). So let $y = mx + b$ be an equation for this line, and let P_3 be the third point where the line intersects the curve E_L . Then $P_1 + P_2 + P_3 = 0$, by the definition of the group law on $E_L(\mathbb{C})$.

Now consider the function $\ell(z) = -\wp'(z) + m\wp(z) + b$. It is an elliptic function of order 3 with a triple pole at 0, so it has three zeros in the fundamental parallelogram \mathcal{F}_0 , two of which are z_1 and z_2 . Let z_3 be the third zero in \mathcal{F}_0 . The point $\Phi(z_3)$ lies on both the line ℓ and the elliptic curve $E_L(\mathbb{C})$, hence it must lie in $\{P_1, P_2, P_3\}$; moreover, we have a bijection from $\{z_1, z_2, z_3\}$ to $\{\Phi(z_1), \Phi(z_2), \Phi(z_3)\} = \{P_1, P_2, P_3\}$, and this bijection must send z_3 to P_3 if P_3 is distinct from P_1 and P_2 . If P_3 coincides with exactly one of P_1 or P_2 , say P_1 , then $\ell(z)$ has a double zero at z_1 and we must have $z_3 = z_1$; and if $P_1 = P_2 = P_3$ then clearly $z_1 = z_2 = z_3$. Thus in every case we must have $P_3 = \Phi(z_3)$.

We have $P_1 + P_2 + P_3 = 0$, so it suffices to show $z_1 + z_2 + z_3 \in L$, since this implies

$$\Phi(z_1 + z_2) = \Phi(-z_3) = -\Phi(z_3) = -P_3 = P_1 + P_2 = \Phi(z_1) + \Phi(z_2).$$

Let \mathcal{F}_α be a fundamental parallelogram for L whose boundary does not contain any zeros or poles of $\ell(z)$ and replace z_1, z_2, z_3 by equivalent points in \mathcal{F}_α if necessary.

Applying Theorem 14.17 to $g(z) = z$ and $f(z) = \ell(z)$ yields

$$\frac{1}{2\pi i} \int_{\partial \mathcal{F}_\alpha} z \frac{\ell'(z)}{\ell(z)} dz = \sum_{w \in F_\alpha} \text{ord}_w(\ell) w = z_1 + z_2 + z_3 - 3 \cdot 0 = z_1 + z_2 + z_3, \quad (1)$$

where the boundary $\partial \mathcal{F}_\alpha$ of \mathcal{F}_α is oriented counter-clockwise.

Let us now evaluate the integral in (1); to ease the notation, define $f(z) := \ell'(z)/\ell(z)$, which we note is an elliptic function (hence periodic with respect to L). We then have

$$\begin{aligned} \int_{\partial \mathcal{F}_\alpha} z f(z) dz &= \int_{\alpha}^{\alpha+\omega_1} z f(z) dz + \int_{\alpha+\omega_1}^{\alpha+\omega_1+\omega_2} z f(z) dz + \int_{\alpha+\omega_1+\omega_2}^{\alpha+\omega_2} z f(z) dz + \int_{\alpha+\omega_2}^{\alpha} z f(z) dz \\ &= \int_{\alpha}^{\alpha+\omega_1} z f(z) dz + \int_{\alpha}^{\alpha+\omega_2} (z + \omega_1) f(z) dz + \int_{\alpha+\omega_1}^{\alpha} (z + \omega_2) f(z) dz + \int_{\alpha+\omega_2}^{\alpha} z f(z) dz \\ &= \omega_1 \int_{\alpha}^{\alpha+\omega_2} f(z) dz + \omega_2 \int_{\alpha+\omega_1}^{\alpha} f(z) dz. \end{aligned} \quad (2)$$

Note that we have used the periodicity of $f(z)$ to replace $f(z + \omega_i)$ by $f(z)$, and to cancel integrals in opposite directions along lines that are equivalent modulo L .

For any closed (not necessarily simple) curve C and a point $z_0 \notin C$, the quantity

$$\frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

is the *winding number* of C about z_0 , and it is an integer (it counts the number of times the curve C “winds around” the point z_0); see [1, Lem. 4.2.1] or [4, Lem. B.1.3].

The function $\ell(\alpha + t\omega_2)$ parametrizes a closed curve C_1 from $\ell(\alpha)$ to $\ell(\alpha + \omega_2) = \ell(\alpha)$, as t ranges from 0 to 1. The winding number of C_1 about the point 0 is the integer

$$c_1 := \frac{1}{2\pi i} \int_{C_1} \frac{dz}{z - 0} = \frac{1}{2\pi i} \int_0^1 \frac{\ell'(\alpha + t\omega_2)}{\ell(\alpha + t\omega_2)} \omega_2 dt = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_2} \frac{\ell'(z)}{\ell(z)} dz = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_2} f(z) dz. \quad (3)$$

Similarly, the function $\ell(\alpha + t\omega_1)$ parameterizes a closed curve C_2 from $\ell(\alpha)$ to $\ell(\alpha + \omega_1)$, and we obtain the integer

$$c_2 := \frac{1}{2\pi i} \int_{C_2} \frac{dz}{z - 0} = \frac{1}{2\pi i} \int_0^1 \frac{\ell'(\alpha + t\omega_1)}{\ell(\alpha + t\omega_1)} \omega_1 dt = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_1} \frac{\ell'(z)}{\ell(z)} dz = \frac{1}{2\pi i} \int_\alpha^{\alpha+\omega_1} f(z) dz. \quad (4)$$

Plugging (3), and (4) into (2), and applying (1), we see that

$$z_1 + z_2 + z_3 = c_1\omega_1 - c_2\omega_2 \in L,$$

as desired. □

15.2 The j -invariant of a lattice

Definition 15.2. The j -invariant of a lattice L is defined by

$$j(L) = 1728 \frac{g_2(L)^3}{\Delta(L)} = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2}.$$

Recall that $\Delta(L) \neq 0$, by Lemma 14.33, so $j(L)$ is always defined.

The elliptic curve $E_L: y^2 = 4x^3 - g_2(L)x - g_3(L)$ is isomorphic to the elliptic curve $y^2 = x^3 + Ax + B$, where $g_2(L) = -4A$ and $g_3(L) = -4B$. Thus

$$j(L) = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = 1728 \frac{(-4A)^3}{(-4A)^3 - 27(-4B)^2} = 1728 \frac{4A^3}{4A^3 + 27B^2} = j(E_L).$$

Thus the j -invariant of a lattice L is the same as the j -invariant of the corresponding elliptic curve E_L . We now define the discriminant of an elliptic curve so that it agrees with the discriminant of the corresponding lattice.

Definition 15.3. The *discriminant* of an elliptic curve $E: y^2 = x^3 + Ax + B$ is

$$\Delta(E) = -16(4A^3 + 27B^2).$$

This definition applies to any elliptic curve E/k defined by a short Weierstrass equation, whether $k = \mathbb{C}$ or not, but for the moment we continue to focus on elliptic curves over \mathbb{C} .

Recall from Theorem 13.14 that elliptic curves E/k and E'/k are isomorphic over \bar{k} if and only if $j(E) = j(E')$. Thus over an algebraically closed field like \mathbb{C} , the j -invariant characterizes elliptic curves up to isomorphism. We now define an analogous notion of isomorphism for lattices.

Definition 15.4. Lattices L and L' are said to be *homothetic* if $L' = \lambda L$ for some $\lambda \in \mathbb{C}^\times$.

Theorem 15.5. *Two lattices L and L' are homothetic if and only if $j(L) = j(L')$.*

Proof. Suppose L and L' are homothetic, with $L' = \lambda L$. Then

$$g_2(L') = 60 \sum_{\omega \in L'^*} \frac{1}{\omega^4} = 60 \sum_{\omega \in L^*} \frac{1}{(\lambda\omega)^4} = \lambda^{-4} g_2(L).$$

Similarly, $g_3(L') = \lambda^{-6} g_3(L)$, and we have

$$j(L') = 1728 \frac{(\lambda^{-4} g_2(L))^3}{(\lambda^{-4} g_2(L))^3 - 27(\lambda^{-6} g_3(L))^2} = 1728 \frac{g_2(L)^3}{g_2(L)^3 - 27g_3(L)^2} = j(L).$$

To show the converse, let us now assume $j(L) = j(L')$. Let E_L and $E_{L'}$ be the corresponding elliptic curves. Then $j(E_L) = j(E_{L'})$. We may write

$$E_L: y^2 = x^3 + Ax + B,$$

with $-4A = g_2(L)$ and $-4B = g_3(L)$, and similarly for $E_{L'}$, with $-4A' = g_2(L')$ and $-4B' = g_3(L')$. By Theorem 13.13, there is a $\mu \in \mathbb{C}^\times$ such that $A' = \mu^4 A$ and $B' = \mu^6 B$, and if we let $\lambda = 1/\mu$, then $g_2(L') = \lambda^{-4} g_2(L) = g_2(\lambda L)$ and $g_3(L') = \lambda^{-6} g_3(L) = g_3(\lambda L)$, as above. We now show that this implies $L' = \lambda L$.

Recall from Theorem 14.29 that the Weierstrass \wp -function satisfies

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3.$$

Differentiating both sides yields

$$\begin{aligned} 2\wp'(z)\wp''(z) &= 12\wp(z)^2\wp'(z) - g_2\wp'(z) \\ \wp''(z) &= 6\wp(z)^2 - \frac{g_2}{2}. \end{aligned} \tag{5}$$

By Theorem 14.28, the Laurent series for $\wp(z; L)$ at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1)G_{2n+2}z^{2n} = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^{2n},$$

where $a_1 = g_2/20$ and $a_2 = g_3/28$.

Comparing coefficients for the z^{2n} term in (5), we find that for $n \geq 2$ we have

$$(2n+2)(2n+1)a_{n+1} = 6 \left(\sum_{k=1}^{n-1} a_k a_{n-k} + 2a_{n+1} \right),$$

and therefore

$$a_{n+1} = \frac{6}{(2n+2)(2n+1) - 12} \sum_{k=1}^{n-1} a_k a_{n-k}.$$

This allows us to compute a_{n+1} from a_1, \dots, a_{n-1} , for all $n \geq 2$. It follows that $g_2(L)$ and $g_3(L)$ uniquely determine the function $\wp(z) = \wp(z; L)$ (and therefore the lattice L where $\wp(z)$ has poles), since $\wp(z)$ is uniquely determined by its Laurent series expansion about 0.

Now consider L' and λL , where we have $g_2(L') = g_2(\lambda L)$ and $g_3(L') = g_3(\lambda L)$. It follows that $\wp(z; L') = \wp(z; \lambda L)$ and $L' = \lambda L$, as desired. \square

Corollary 15.6. *Two lattices L and L' are homothetic if and only if the corresponding elliptic curves E_L and $E_{L'}$ are isomorphic.*

Thus homothety classes of lattices correspond to isomorphism classes of elliptic curves over \mathbb{C} , and both are classified by the j -invariant. Recall from Theorem 13.12 that every complex number is the j -invariant of an elliptic curve E/\mathbb{C} . To prove the Uniformization Theorem we just need to show that the same is true of lattices.

15.3 The j -function

Every lattice $[\omega_1, \omega_2]$ is homothetic to a lattice of the form $[1, \tau]$, with τ in the upper half plane $\mathcal{H} := \{z \in \mathbb{C} : \text{im } z > 0\}$; we may take $\tau = \pm\omega_2/\omega_1$ with the sign chosen so that $\text{im } \tau > 0$. This leads to the following definition of the j -function.

Definition 15.7. The j -function $j: \mathcal{H} \rightarrow \mathbb{C}$ is defined by $j(\tau) = j([1, \tau])$. We similarly define $g_2(\tau) = g_2([1, \tau])$, $g_3(\tau) = g_3([1, \tau])$, and $\Delta(\tau) = \Delta([1, \tau])$.

Note that for any $\tau \in \mathcal{H}$, both $-1/\tau$ and $\tau + 1$ lie in \mathcal{H} (the maps $\tau \mapsto 1/\tau$ and $\tau \mapsto -\tau$ both swap the upper and lower half-planes; their composition preserves them).

Theorem 15.8. *The j -function is holomorphic on \mathcal{H} , and satisfies $j(-1/\tau) = j(\tau)$ and $j(\tau + 1) = j(\tau)$.*

Proof. From the definition of $j(\tau) = j([1, \tau])$ we have

$$j(\tau) = 1728 \frac{g_2(\tau)^3}{\Delta(\tau)} = 1728 \frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2}.$$

The series defining

$$g_2(\tau) = 60 \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^4} \quad \text{and} \quad g_3(\tau) = 140 \sum_{\substack{m, n \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^6}$$

converge absolutely for any fixed $\tau \in \mathcal{H}$, by Lemma 14.22, and they converge uniformly over τ in any compact subset of \mathcal{H} . The proof of this last fact is straight-forward but slightly technical; see [2, Thm. 1.15] for the details. It follows that $g_2(\tau)$ and $g_3(\tau)$ are holomorphic on \mathcal{H} , and therefore $\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2$ is also holomorphic on \mathcal{H} . Since $\Delta(\tau)$ is nonzero for all $\tau \in \mathcal{H}$, by Lemma 14.33, the j -function $j(\tau)$ is holomorphic on \mathcal{H} as well.

The lattices $[1, \tau]$ and $[1, -1/\tau] = -1/\tau[1, \tau]$ are homothetic, and the lattices $[1, \tau + 1]$ and $[1, \tau]$ are equal; thus $j(-1/\tau) = j(\tau)$ and $j(\tau + 1) = j(\tau)$, by Theorem 15.5. \square

15.4 The modular group

We now consider the *modular group*

$$\Gamma = \text{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

As proved in Problem Set 8, the group Γ acts on \mathcal{H} via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d},$$

and it is generated by the matrices $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. This implies that the j -function is invariant under the action of the modular group; in fact, more is true.

Lemma 15.9. For any $\tau, \tau' \in \mathcal{H}$ we have $j(\tau) = j(\tau')$ if and only if $\tau' = \gamma\tau$ for some $\gamma \in \Gamma$.

Proof. We have $j(S\tau) = j(-1/\tau) = j(\tau)$ and $j(T\tau) = j(\tau + 1) = j(\tau)$, by Theorem 15.8. It follows that if $\tau' = \gamma\tau$ then $j(\tau') = j(\tau)$, since S and T generate Γ .

To prove the converse, let us suppose that $j(\tau) = j(\tau')$. Then by Theorem 15.5, the lattices $[1, \tau]$ and $[1, \tau']$ are homothetic. So $[1, \tau'] = \lambda[1, \tau]$, for some $\lambda \in \mathbb{C}^\times$. There thus exist integers a, b, c , and d such that

$$\begin{aligned}\tau' &= a\lambda\tau + b\lambda \\ 1 &= c\lambda\tau + d\lambda\end{aligned}$$

From the second equation, we see that $\lambda = \frac{1}{c\tau + d}$. Substituting this into the first, we have

$$\tau' = \frac{a\tau + b}{c\tau + d} = \gamma\tau, \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2 \times 2}.$$

Now $[1, \tau'] = \lambda[1, \tau]$ implies $\text{im } \tau' = |\lambda|^2 \text{im } \tau$, since $\tau, \tau' \in \mathcal{H}$ and fundamental parallelograms for $[1, \tau'] = \lambda[1, \tau]$ must have the same area. But we also have

$$\text{im } \tau' = \text{im}(\gamma\tau) = \text{im} \left(\frac{a\tau + b}{c\tau + d} \right) = \frac{\text{im}((a\tau + b)(c\bar{\tau} + d))}{|c\tau + d|^2} = \frac{(ad - bc) \text{im } \tau}{|c\tau + d|^2} = (\det \gamma) |\lambda|^2 \text{im } \tau,$$

and therefore $\det \gamma = 1$ and $\gamma \in \text{SL}_2(\mathbb{Z})$. □

Lemma 15.9 implies that when studying the j -function it suffices to study its behavior on Γ -equivalence classes of \mathcal{H} , that is, the orbits of \mathcal{H} under the action of Γ . We thus consider the quotient of \mathcal{H} modulo Γ -equivalence, which we denote by \mathcal{H}/Γ .² The actions of γ and $-\gamma$ are identical, so taking the quotient by $\text{PSL}_2(\mathbb{Z}) = \text{SL}_2(\mathbb{Z})/\{\pm 1\}$ yields the same result, but for the sake of clarity we will stick with $\Gamma = \text{SL}_2(\mathbb{Z})$.

We now wish to determine a fundamental domain for \mathcal{H}/Γ , a set of unique representatives in \mathcal{H} for each Γ -equivalence class. For this purpose we will use the set

$$\mathcal{F} = \{\tau \in \mathcal{H} : \text{re}(\tau) \in [-1/2, 1/2) \text{ and } |\tau| \geq 1, \text{ such that } |\tau| > 1 \text{ if } \text{re}(\tau) > 0\}.$$

Lemma 15.10. The set \mathcal{F} is a fundamental domain for \mathcal{H}/Γ .

Proof. We need to show that for every $\tau \in \mathcal{H}$, there is a unique $\tau' \in \mathcal{F}$ such that $\tau' = \gamma\tau$, for some $\gamma \in \Gamma$. We first prove existence. Let us fix $\tau \in \mathcal{H}$. For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$\text{im}(\gamma\tau) = \text{im} \left(\frac{a\tau + b}{c\tau + d} \right) = \frac{\text{im}((a\tau + b)(c\bar{\tau} + d))}{|c\tau + d|^2} = \frac{(ad - bc) \text{im } \tau}{|c\tau + d|^2} = \frac{\text{im } \tau}{|c\tau + d|^2} \quad (6)$$

Let $c\tau + d$ be a shortest vector in the lattice $[1, \tau]$. Then c and d must be relatively prime, and we can pick integers a and b so that $ad - bc = 1$. The matrix $\gamma_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then maximizes the value of $\text{im}(\gamma\tau)$ over $\gamma \in \Gamma$. Let us now choose $\gamma = T^k \gamma_0$, where k is chosen so that $\text{re}(\gamma\tau) \in [1/2, 1/2)$, and note that $\text{im}(\gamma\tau) = \text{im}(\gamma_0\tau)$ remains maximal. We must have $|\gamma\tau| \geq 1$, since otherwise $\text{im}(S\gamma\tau) > \text{im}(\gamma\tau)$, contradicting the maximality of $\text{im}(\gamma\tau)$.

²Some authors write this quotient as $\Gamma \backslash \mathcal{H}$ to indicate that the action is on the left.

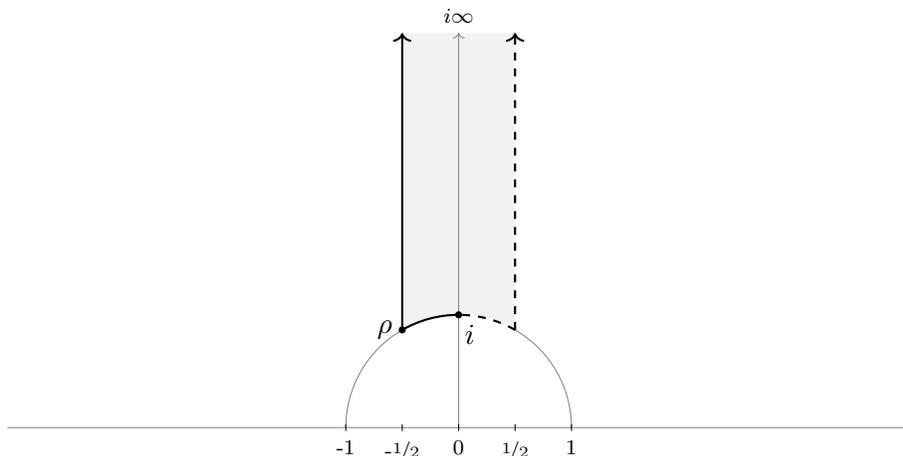


Figure 1: Fundamental domain \mathcal{F} for \mathcal{H}/Γ , with $i = e^{\pi i/2}$ and $\rho = e^{2\pi i/3}$.

Finally, if $\tau' = \gamma\tau \notin \mathcal{F}$, then we must have $|\gamma\tau| = 1$ and $\operatorname{re}(\gamma\tau) > 0$, in which case we replace γ by $S\gamma$ so that $\tau' = \gamma\tau \in \mathcal{F}$.

It remains to show that τ' is unique. This is equivalent to showing that any two Γ -equivalent points in \mathcal{F} must coincide. So let τ_1 and $\tau_2 = \gamma_1\tau_1$ be two elements of \mathcal{F} , with $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and assume $\operatorname{im} \tau_1 \leq \operatorname{im} \tau_2$. By (6), we must have $|c\tau_1 + d|^2 \leq 1$, thus

$$1 \geq |c\tau_1 + d|^2 = (c\tau_1 + d)(c\bar{\tau}_1 + d) = c^2|\tau_1|^2 + d^2 + 2cd \operatorname{re} \tau_1 \geq c^2|\tau_1|^2 + d^2 - |cd| \geq 1,$$

where the last inequality follows from $|\tau_1| \geq 1$ and the fact that c and d cannot both be zero (since $\det \gamma = 1$). Thus $|c\tau_1 + d| = 1$, which implies $\operatorname{im} \tau_2 = \operatorname{im} \tau_1$. We also have $|c|, |d| \leq 1$, and by replacing γ_1 by $-\gamma_1$ if necessary, we may assume that $c \geq 0$. This leaves 3 cases:

1. $c = 0$: then $|d| = 1$ and $a = d$. So $\tau_2 = \tau_1 \pm b$, but $|\operatorname{re} \tau_2 - \operatorname{re} \tau_1| < 1$, so $\tau_2 = \tau_1$.
2. $c = 1, d = 0$: then $b = -1$ and $|\tau_1| = 1$. So τ_1 is on the unit circle and $\tau_2 = a - 1/\tau_1$. Either $a = 0$ and $\tau_2 = \tau_1 = i$, or $a = -1$ and $\tau_2 = \tau_1 = \rho$.
3. $c = 1, |d| = 1$: then $|\tau_1 + d| = 1$, so $\tau_1 = \rho$, and $\operatorname{im} \tau_2 = \operatorname{im} \tau_1 = \sqrt{3}/2$ implies $\tau_2 = \rho$.

In every case we have $\tau_1 = \tau_2$ as desired. \square

Theorem 15.11. *The restriction of the j -function to \mathcal{F} defines a bijection from \mathcal{F} to \mathbb{C} .*

Proof. Injectivity follows immediately from Lemmas 15.9 and 15.10. It remains to prove surjectivity. We have

$$g_2(\tau) = 60 \sum_{\substack{n, m \in \mathbb{Z} \\ (m, n) \neq (0, 0)}} \frac{1}{(m + n\tau)^4} = 60 \left(2 \sum_{m=1}^{\infty} \frac{1}{m^4} + \sum_{\substack{n, m \in \mathbb{Z} \\ n \neq 0}} \frac{1}{(m + n\tau)^4} \right).$$

The second sum tends to 0 as $\operatorname{im} \tau \rightarrow \infty$. Thus we have

$$\lim_{\operatorname{im} \tau \rightarrow \infty} g_2(\tau) = 120 \sum_{m=1}^{\infty} m^{-4} = 120 \zeta(4) = 120 \frac{\pi^4}{90} = \frac{4\pi^4}{3},$$

where $\zeta(s)$ is the Riemann zeta function. Similarly,

$$\lim_{\text{im}\tau \rightarrow \infty} g_3(\tau) = 280 \zeta(6) = 280 \frac{\pi^6}{945} = \frac{8\pi^6}{27}.$$

Thus

$$\lim_{\text{im}\tau \rightarrow \infty} \Delta(\tau) = \left(\frac{4}{3}\pi^4\right)^3 - 27 \left(\frac{8}{27}\pi^6\right)^2 = 0.$$

(this explains the coefficients 60 and 140 in the definitions of g_2 and g_3 ; they are the smallest pair of integers that ensure this limit is 0). Since $\Delta(\tau)$ is the denominator of $j(\tau)$, the quantity $j(\tau) = 1728g_2(\tau)^3/\Delta(\tau)$ is unbounded as $\text{im}\tau \rightarrow \infty$.

In particular, the j -function is non-constant, and by Theorem 15.8 it is holomorphic on \mathcal{H} . The open mapping theorem implies that $j(\mathcal{H})$ is an open subset of \mathbb{C} ; see [4, Thm. 3.4.4].

We claim that $j(\mathcal{H})$ is also a closed subset of \mathbb{C} . Let $j(\tau_1), j(\tau_2), \dots$ be an arbitrary convergent sequence in $j(\mathcal{H})$, converging to $w \in \mathbb{C}$. The j -function is Γ -invariant, by Lemma 15.9, so we may assume the τ_n all lie in \mathcal{F} . The sequence $\text{im}\tau_1, \text{im}\tau_2, \dots$ must be bounded, say by B , since $j(\tau) \rightarrow \infty$ as $\text{im}\tau \rightarrow \infty$, but the sequence $j(\tau_1), j(\tau_2), \dots$ converges; it follows that the τ_n all lie in the compact set

$$\Omega = \{\tau : \text{re}\tau \in [-1/2, 1/2], \text{im}\tau \in [1/2, B]\}.$$

There is thus a subsequence of the τ_n that converges to some $\tau \in \Omega \subset \mathcal{H}$. The j -function is holomorphic, hence continuous, so $j(\tau) = w$. It follows that the open set $j(\mathcal{H})$ contains all its limit points and is therefore closed.

The fact that the non-empty set $j(\mathcal{H}) \subseteq \mathbb{C}$ is both open and closed implies that $j(\mathcal{H}) = \mathbb{C}$, since \mathbb{C} is connected. It follows that $j(\mathcal{F}) = \mathbb{C}$, since every element of \mathcal{H} is Γ -equivalent to an element of \mathcal{F} (Lemma 15.10) and the j -function is Γ -invariant (Lemma 15.9). \square

Corollary 15.12 (Uniformization Theorem). *For every elliptic curve E/\mathbb{C} there exists a lattice L such that $E = E_L$.*

Proof. Given E/\mathbb{C} , pick $\tau \in \mathcal{H}$ so that $j(\tau) = j(E)$ and let $L' = [1, \tau]$. We have

$$j(E) = j(\tau) = j(L') = j(E_{L'}),$$

so E is isomorphic to $E_{L'}$, by Theorem 13.13, where the isomorphism is given by the map $(x, y) \mapsto (\mu^2x, \mu^3y)$ for some $\mu \in \mathbb{C}^\times$. If we now let $L = \frac{1}{\mu}L'$, then $E = E_L$. \square

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