2.001 - MECHANICS AND MATERIALS I
Lecture #23
11/29/2006
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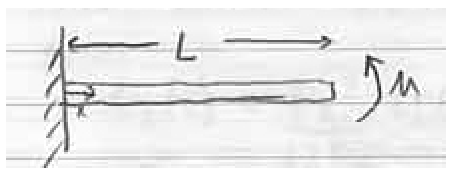
Recall Moment-Curvature Equation

$$M(x) = \frac{E(x)I(x)}{\rho(x)} \text{ or } EI_{eff} \text{ for composite beams.}$$

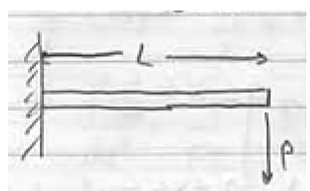
$$\frac{1}{\rho(x)} = \frac{\partial^2 v}{\partial x^2} = \frac{M(x)}{E(x)I(x)}$$

Approach: Integrate, get $v(x)$, $\theta(x) = \frac{dv(x)}{dx}$. Use boundary conditions to get constants of integration.

Library of Solutions (Thus Far):



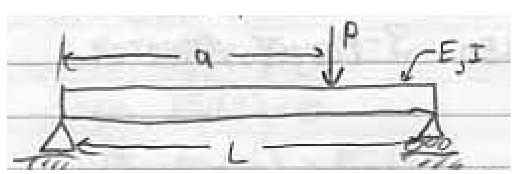
$$v(x) = \frac{Mx^2}{2EI}$$



$$v(x) = \frac{-Px^2}{2EI}\left(L - \frac{x}{3}\right)$$

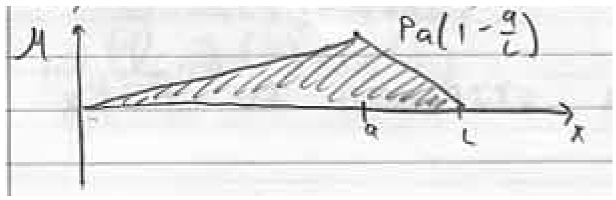
$$v_{tip}(x = L) = \frac{-PL^3}{3EI}$$

Recall example from last time:



$$M(x) = P\left(1 - \frac{a}{L}\right)x, 0 \leq x \leq a$$

$$M(x) = P\left(1 - \frac{a}{L}\right)x - P\left(x - a\right), a \leq x \leq L$$



For $a = \frac{L}{2}$ (symmetric special case)

$$M(x) = \frac{Px}{2}, 0 \leq x \leq \frac{L}{2}$$

$$M(x) = \frac{-Px}{2} + \frac{PL}{2}\frac{L}{2} \leq x \leq L$$

Left:

$$\frac{d^2v}{dx^2} = \frac{Px}{2EI}$$

$$\frac{dv}{dx} = \frac{Px^2}{4EI} + c_1$$

$$v = \frac{Px^3}{12EI} + c_1x + c_2$$

Boundary Conditions: $v(x = 0) = 0 \Rightarrow c_2 = 0$

$\theta(x = \frac{L}{2}) = 0$

$$0 = \frac{P}{4EI}\left(\frac{L}{2}\right)^2 + c_1 \Rightarrow c_1 = \frac{-PL^2}{16EI}$$

So:

Left:

$$v(x) = \frac{P}{12EI}x^3 - \frac{PL^2}{16EI}x, 0 \leq x \leq \frac{L}{2}$$

Right:

$$v(x) = \frac{P}{12EI}(L-x)^3 - \frac{PL^2}{16EI}(L-x), 0 \leq x \leq \frac{L}{2}$$

Note: The right is the same as the left but starting at $x = L$ and moving left. This is due to symmetry. This situation is called 3-point bending.

One can define a stiffness in "F=kx" type equation.

Find v_{max}

$$v\left(\frac{L}{2}\right) = \frac{P}{EI}\left[\frac{1}{12}\left(\frac{L}{2}\right)^3 - \frac{1}{16}\left(\frac{L}{2}\right)L^2\right]$$

$$v_{max} = \frac{PL^3}{EI}\left[\frac{1}{3} - 1\right]$$

$$v_{max} = \frac{-PL^3}{48EI}$$

$F = kx$

$-P = kv_{max}$

So: $k = \frac{48EI}{L^3}$

Now solve again without using symmetry:

Recall:

$$v_L(x) = \frac{Px^3}{12EI} + c_1x + c_2, 0 \leq x \leq \frac{L}{2}$$

Recall:

$$M_z(x) = \frac{-Px}{2} + \frac{PL}{2}, \frac{L}{2} \leq x \leq L$$

$$\frac{d^2v_r}{dx^2} = \frac{1}{EI}\left[\frac{-Px}{2} + \frac{PL}{2}\right]$$

$$\theta_R(x) = \frac{dv_r}{dx} = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right] + c_3$$

$$v_r = \frac{1}{EI}\left[\frac{-Px^3}{12} + \frac{PLx^2}{4}\right] + c_3x + c_4$$

Boundary Conditions

1. $v_L(0) = 0$
2. $v_R(L) = 0$
3. $v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$
4. $\theta_L\left(\frac{L}{2}\right) = \theta_R\left(\frac{L}{2}\right)$

Use 1.

$$v_L(0) = 0$$

$$c_2 = 0$$

Use 2.

$$v_R(L) = 0$$

$$\frac{P}{EI}\left[\frac{-L^3}{12} + \frac{L^3}{4}\right] + c_3L + c_4$$

$$c_3L + c_4 = \frac{-PL^3}{6EI}$$

Use 3.

$$v_L\left(\frac{L}{2}\right) = v_R\left(\frac{L}{2}\right)$$

$$\frac{P}{EI}\left[\frac{1}{12}\left(\frac{L}{2}\right)^3\right] + c_1\frac{L}{2} = \frac{P}{EI}\left[\frac{-1}{12}\left(\frac{L}{2}\right)^3 + \frac{L^3}{16}\right] + c_3\frac{L}{2} + c_4$$

$$(c_3 - c_1)\frac{L}{2} + c_4 = \frac{PL^3}{EI}\left(\frac{-1}{24}\right)$$

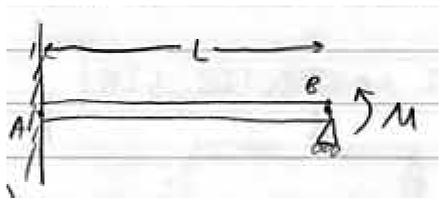
Use 4.

$$\theta_L\left(\frac{L}{2}\right) = \theta_R\left(\frac{L}{2}\right)$$

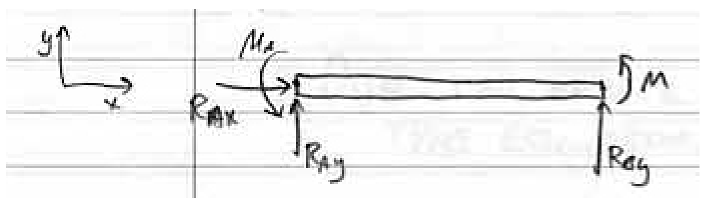
$$c_3 - c_1 = \frac{-PL^2}{8EI}$$

Solve for c_1,c_2, c_3.
Note: Applying symmetry was easier.

Example: Statically Indeterminate

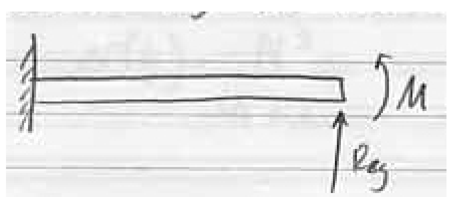


FBD



Solve using superposition

1. Pretend R_{By} is known.



2. Find $v(x)$ and v_{tip}.

$$v(x) = \frac{M}{2EI}x^2 + \frac{R_{By}x^2}{2EI}\left(L - \frac{x}{3}\right)$$

$$v_{tip} = \frac{ML^2}{2EI} + \frac{R_{By}L^3}{3EI}$$

3. Note $v_{tip} = 0$ due to support. This is an additional boundary condition.

$$\frac{ML^2}{2EI} + \frac{R_{By}L^3}{3EI} = 0$$

$$R_{By} = \frac{-3}{2}\frac{M}{L}$$

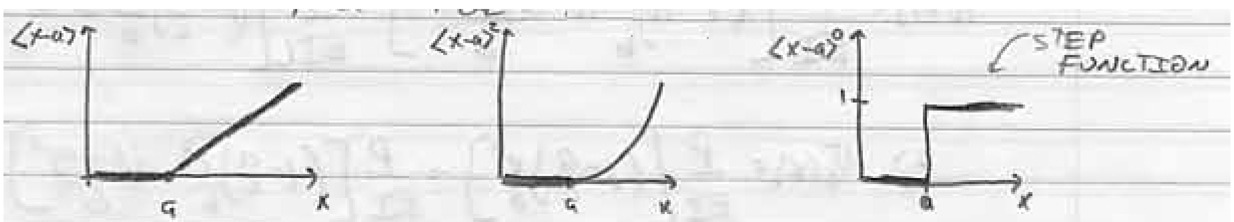
4. Solve for $v(x)$.

$$v(x) = \frac{M}{2EI}x^2 - \frac{3}{2}\frac{M}{L}\left(\frac{x^2}{2EI}\right)\left(L - \frac{x}{3}\right)$$

Discontinuity Functions

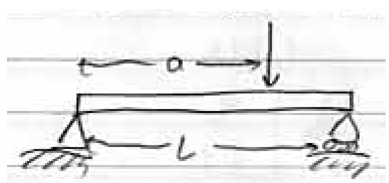
$$< x - a > \equiv 0 \text{ for } x - a < 0$$

$$< x - a > \equiv x - z \text{ for } x - a > 0$$



$$\int < x - a >^n \, dx = \frac{< x - a >^{n+1}}{n+1} + c$$

So recall example:



Rewrite moment equation.

$$M_z(x) = P\left(1 - \frac{a}{L}\right)x - P < x - a >$$

$$\frac{d^2v(x)}{dx^2} = \frac{1}{EI}\left[P\left(1 - \frac{a}{L}\right)x - P < x - a >\right]$$

$$\frac{dv(x)}{dx} = \frac{1}{EI}\left[P\left(1 - \frac{a}{L}\right)x - P < x - a > \right] + c_1$$

$$v(x) = \frac{1}{EI}\left[P\left(1 - \frac{a}{L}\frac{x^3}{6} - P\frac{< x - a >^3}{6}\right] + c_1 x + c_2$$

Boundary Conditions:

$v(0) = 0 \Rightarrow c_2 = 0$

$v(L) = 0 \Rightarrow c_1 = \frac{-P}{EIL}\left[\left(1 - \frac{a}{L}\right)\frac{L^3}{6} - \frac{(L-a)^3}{6}\right.$

So:

$$v(x) = \frac{P}{EI}\left[\left(1 - \frac{a}{L}\right)\frac{x^3}{6} - \frac{< x - a >^3}{6}\right] - \frac{P}{EIL}\left[\left(1 - \frac{a}{L}\right)\frac{L^3}{6} - \frac{(L-a)^3}{6}\right]x$$

So:

$$v_L(x) = \frac{P}{EI}\left[\left(1 - \frac{a}{L}\right)\frac{x^3}{6}\right] - \frac{P}{EI}\left[\left(1 - \frac{a}{L}\right)\frac{L^3}{6} - \frac{(L-a)^3}{6}\right]\frac{x}{L}$$

$$v_R(x) = \frac{P}{EI}\left[\left(1 - \frac{a}{L}\right)\frac{x^3}{6} - \frac{(x-a)^3}{6}\right] - \frac{P}{EI}\left[\left(1 - \frac{a}{L}\right)\frac{L^3}{6} - \frac{(L-a)^3}{6}\right]\frac{x}{L}$$

Check answer. Try $a = \frac{L}{2}$ and compare to earlier result.

$$v\left(\frac{L}{2}\right) = \frac{-PL^3}{48EI}$$