Massachusetts Institute of Technology

8.223, Classical Mechanics II IAP 2017

Solutions 1

1. Consider the motion of an object close to the surface of the Earth moving under the influence of Earth's gravity.

a) Show that the gravitational force law

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

reduces to

$$\vec{F} = -mg\hat{z}$$

at the surface with z pointing toward the zenith. Compute g in SI units.

Solutions:

A cannonball has a range of a few kilometers height (h) and the radius of the earth $R_\oplus = 6400$ km.

$$\vec{F} = -\frac{GM_\oplus m}{(R_\oplus + h)^2}\hat{r} = -\frac{GM_\oplus m}{R_\oplus^2(1+\frac{h}{R_\oplus})^2}\hat{r}$$

By using $\frac{1}{(1+\epsilon)^2} \approx 1 - 2\epsilon$ ($\epsilon << 1$), we get

$$\vec{F} \approx -\frac{GM_\oplus m}{R_\oplus^2}(1 - 2\frac{h}{R_\oplus})\hat{r}$$

Since $\dfrac{h}{R_\oplus} \approx \dfrac{3 \text{ km}}{6400 \text{ km}} \approx 0.5 \times 10^{-3}$, which is in the order of 10^{-3} for a cannonball, we can just ignore the second term in the bracket.

$$\vec{F} \approx -\frac{GM_\oplus m}{R_\oplus^2}\hat{z} \approx -\left(\frac{7\times 10^{-11} \text{ m}^3}{\text{kg}\cdot\text{s}^2}\frac{6\times 10^{24} \text{ kg}}{(6.4\times 10^6 \text{ m})^2}\right)m\hat{z} \approx -(10 \text{ meters}\cdot\text{second}^{-2})m\hat{z}$$

And thus

$$g \approx 10 \text{ meters/second}^2$$

b) For a cannonball fired at velocity v_o and angle θ above the horizon, find the distance between the origin to where the cannonball hits the surface of the Moon.

Solutions:

$$\vec{x}(t) = (v_0\cos\theta)t\hat{x} + (v_0\sin\theta)t\hat{z} - \frac{1}{2}g_{\text{Moon}}t^2\hat{z}$$

When the cannonball hits the surface of the Moon,

$$z(\tau) = 0 = (v_0\sin\theta)\tau - \frac{1}{2}g_{\text{Moon}}\tau^2$$

$$\tau = 0 \quad \text{or} \quad \tau = \frac{2v_0 \sin\theta}{g_{\text{Moon}}}$$

The horizontal distance the ball travels is

$$x(\tau) = (v_0 \cos\theta)\tau = \frac{(v_0 \cos\theta)(2v_0 \sin\theta)}{g_{\text{Moon}}} = \frac{v_0^2 \sin 2\theta}{g_{\text{Moon}}}$$

We can compute a numerical value for g_{Moon} as well

$$g_{\text{Moon}} = \frac{GM_{\text{Moon}}}{R_{\text{Moon}}^2} \approx \frac{7 \times 10^{-11}\ \text{m}^3}{\text{kg} \cdot \text{s}^2} \frac{7.3 \times 10^{22}\text{kg}}{(1.7 \times 10^6\text{m})^2} \approx 1.7\text{m} \cdot \text{s}^{-2}$$

2. Hooke's law for a spring of constant k is $F = -kx$. A mass m is pushed from position x_o with velocity v_o at $t = 0$. Find the subsequent motion, $x(t)$.

   Solutions:

   The equation of motion for this system is

   $$F = -kx = m\ddot{x}$$

   And the solution is in the format of

   $$x(t) = A \sin\omega t + B \cos\omega t$$

   where $\omega = \sqrt{\dfrac{k}{m}}$. The original position is

   $$x(0) = B = x_0$$

   The original velocity is

   $$\dot{x}(0) = A\omega = v_0$$

   Thus,

   $$x(t) = \frac{v_0}{\omega} \sin\omega t + x_0 \cos\omega t$$

3. Some exercises with coordinate systems

   a) Find the conversion from polar to cartesian coordinates, i.e. the functions $x(r, \theta, \phi)$, $y(r, \theta, \phi)$ and $z(r, \theta, \phi)$. Also find the inverse functions, $r(x, y, z)$, etc.

   Solutions:

   Polar to cartesian:

   $$x(r, \theta, \phi) = r \sin\theta \cos\phi$$
   $$y(r, \theta, \phi) = r \sin\theta \sin\phi$$
   $$z(r, \theta, \phi) = r \cos\theta$$

   Inversely,

   $$r(x, y, x) = \sqrt{x^2 + y^2 + z^2}$$
   $$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \text{or} \quad \tan\theta = \frac{z}{\sqrt{x^2 + y^2}}$$
   $$\tan\phi = \frac{y}{x}$$

b) Find the conversion from cylindrical to cartesian coordinates, i.e. the functions $x(\rho, \phi, z)$, etc. Also find the inverse functions, $\rho(x, y, z)$, etc.

Solutions:

Cylindrical to cartesian:

$$x(\rho, \phi, z) = \rho \cos \phi$$
$$y(\rho, \phi, z) = \rho \sin \phi$$
$$z(\rho, \phi, z) = z$$

Inversely,

$$\rho(x, y, z) = \sqrt{x^2 + y^2}$$
$$\tan \phi = \frac{y}{x}$$
$$z = z$$

c) A particle starting from the origin moves in the $\hat{r}$ direction with velocity v_o and polar angles ϕ_0 and θ_0, such that $\vec{v} = v_o \hat{r}$. Express $\vec{v}$ in cartesian coordinates.

Solutions:

$$\vec{v} = v_o \hat{r} = v_0 \sin \theta_0 \cos \phi_0 \hat{x} + v_0 \sin \theta_0 \sin \phi_0 \hat{y} + v_0 \cos \theta_0 \hat{z}$$

d) OPTIONAL: A particle starting from a point $\vec{x}_o$ moves with the same velocity from part c) above (now no longer in the $\hat{r}$ direction). Express $\vec{v}$ in polar coordinates. From this exercise, you should notice a BIG weakness of the polar coordinate system.

Solutions:

The new position of the particle can be expressed as

$$\vec{\rho} = \vec{x_0} + \vec{v}t = (r_0 \sin \Theta_0 \cos \Phi_0 + v_0 \sin \theta_0 \cos \phi_0 \, t)\hat{x} + (r_0 \sin \Theta_0 \sin \Phi_0 + v_0 \sin \theta_0 \sin \phi_0 \, t)\hat{y} + (r_0 \cos \Theta_0 + v_0 \cos \theta_0 \, t)\hat{z}$$

$$\rho = \sqrt{(r_0 \sin \Theta_0 \cos \Phi_0 + v_0 \sin \theta_0 \cos \phi_0 \, t)^2 + (r_0 \sin \Theta_0 \sin \Phi_0 + v_0 \sin \theta_0 \sin \phi_0 \, t)^2 + (r_0 \cos \Theta_0 + v_0 \cos \theta_0 \, t)^2}$$

where Θ_0 and Φ_0 are the two polar angles corresponding to $\vec{x}_0$.

Then the new polar angles of $\vec{v}$ relative to the origin are,

$$\cos \theta = \frac{r_0 \cos \Theta_0 + v_0 \cos \theta_0 \, t}{\rho}$$

$$\tan \phi = \frac{r_0 \sin \Theta_0 \sin \Phi_0 + v_0 \sin \theta_0 \sin \phi_0 \, t}{r_0 \sin \Theta_0 \cos \Phi_0 + v_0 \sin \theta_0 \cos \phi_0 \, t}$$

4. Consider the equation of motion for a pendulum swinging under the influence of Earth's gravity

$$ml^2 \ddot{\phi} = -mgl \sin \phi$$

a) Derive this equation of motion (using the Euler-Lagrange equation), with $\phi = 0$ corresponding to the y-axis as in class. Recall that we implement the constraint of a rigid pendulum rod as $x = \ell \sin(\phi)$, $y = \ell \cos(\phi)$

Solutions:

From

$$x(t) = l \sin \phi$$
$$y(t) = l \cos \phi$$

we obtain

$$\dot{x} = l\cos\phi\,\dot{\phi}$$
$$\dot{y} = -l\sin\phi\,\dot{\phi}$$

And the Lagrangian is

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - (-mgy) = \frac{1}{2}ml^2\dot{\phi}^2 + mgl\cos\phi$$

Remember, y is *parallel* to gravity, so it points down, and the potential energy is $V = -mgy$. Feeding this into the Euler-Lagrange equation yields

$$\frac{d}{dt}(ml^2\dot{\phi}) + mgl\sin\phi = mgl^2\ddot{\phi} + mgl\sin\phi = 0$$

b) Justify that for small oscillations, we may use $\sin\phi \sim \phi$.

Solutions:

For small oscillations,

$$\sin\phi \approx \phi - \frac{\phi^3}{3!} \approx \phi$$

if ϕ is small.

$$\ddot{\phi} \approx -\frac{g}{l}\phi$$

c) Solve the resulting equation for arbitrary (but small) initial conditions to give $\phi(t)$.

Solutions:

By solving the equation of motion in small osillations, we obtain

$$\phi(t) = \phi_0\cos\omega t + \frac{\dot{\phi}_0}{\omega}\sin\omega t$$

where $\omega = \sqrt{\dfrac{g}{l}}$.

5. Show that the n Euler-Lagrange equations $\dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} = 0$ hold for $L(q_1, \cdots, q_n, \dot{q}_1, \cdots, q_n)$.

   Solutions:

   Take $L = L(q_1, q_2, \dot{q}_1, \dot{q}_2)$ for example, vary $q_1 + \delta q_1$ and $q_2 + \delta q_2$, then, by following the steps in Landau Chapter 1 Page 3 (essentially integration by parts),

$$\delta S = \int_{t_1}^{t_2}\left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\delta q_1 + \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}\right)\delta q_2\right]dt$$

   Using $\delta q_2 = 0$ and arbitrary $\delta q_1 \neq 0$, we get

$$\delta S = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}\right)\delta q_1 dt = 0$$

   for arbitrary δq_1.

$$\Rightarrow \frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = 0$$

   In a similar way, if we fix $\delta q_1 = 0$, we would get

$$\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = 0$$

   It is clear that if we expand this to an arbitrary number of degrees of freedom, setting all $\delta q_{j\neq i} = 0$ and analyzing the equations for arbitrary δq_i yields the Euler-Lagrange equations for all i.

6. Referring to Eq. 2.8 and the integral equations which follows in Landau,

   a) explain in words why S and S' "differ by a quantity which gives zero on variation"

   Solutions:

   Adding a total time derivative to the Lagrangian adds a constant to the action, and the variation of the constant will vanish.

$$L' = L + \frac{d}{dt}f \Rightarrow S' = \int_0^1 dtL' = \int_0^1 dt\left(L + \frac{d}{dt}f\right) = S + f(1) - f(0) = S + \text{constant}$$

   b) give 2 example functions which can be written as a total time derivative. That is, find 2 esamples of $g(t)$ such that

$$g(q, \dot{q}, t) = \frac{d}{dt}f(q, t)$$

   Solutions:

   There are many possible examples, but a few are

   - any constant : $g(q, \dot{q}, t) = c$
   - any function of t alone : $g(q, \dot{q}, t) = g(t)$
   - any function of q alone times $\dot{q}$: $g(q, \dot{q}, t) = g(q)\dot{q}$

   c) five 2 example functions which **cannot** be written as a total time derivative (i.e., find 2 examples of $g(q, \dot{q}, t)$ such that no $f(q, t)$ can be fond for the equation in part b).

   Again, there are many possible examples, but a few are

   - any function of q times a high-order monomial of $\dot{q}$: $g(q, \dot{q}, t) = g(q)\dot{q}^n \mid n > 1$
   - any function of time times $\dot{q}$: $g(q, \dot{q}, t) = g(t)\dot{q}$

7. For a free particle, we showed that $L = L(v^2)$ if the particle moves in some direction with velocity v. Carry out a Galilean transformation to a primed frame moving with velocity $\vec{u}$ with respect to the original (unprimed) frame

$$\vec{x}' = \vec{x} - \vec{u}t, t' = t$$

   and show that if the unprimed frame is inertial, the primed frame is as well. Hint: you can take the explicit form of the Lagrangian for a free particle $L = \frac{1}{2}mv^2$, and show that L in the unprimed frame and L in the primved frame (i.e. $L' = L((\vec{u} - \vec{v})^2)$) give the same equation of motion.

   Solutions:

   In the unprimed frame, $L = L(v^2)$, then $\frac{\partial L}{\partial x_i} = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$. In the primed frame,

$$\vec{v}' = \vec{v} - \vec{u}$$

$$\vec{x}' = \vec{x} - \vec{u}t$$

$$L((\vec{v} - \vec{u})^2) = L(v^2 - 2\vec{v} \cdot \vec{u} + u^2)$$

   First following the steps in Landau Chapter 1 Page 6, the Lagrangian takes the form of

$$L(v^2) = \frac{1}{2}mv^2$$

   Thus,

$$L' = L((\vec{v} - \vec{u})^2) = \frac{1}{2}mv'^2 = \frac{1}{2}m(v^2 - 2\vec{v} \cdot \vec{u} + u^2) = \frac{1}{2}mv^2 + m\frac{d}{dt}(-\vec{x} \cdot \vec{u} + 2u^2t)$$

$$L' = L + m\frac{d}{dt}(-\vec{x}\cdot\vec{u} + 2u^2 t)$$

The second term is a total time derivative and may be omitted.

Alternatively, the Euler-Lagrange equations for both Lagrangians yield the same equations of motion

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}L\right) - \frac{\partial}{\partial q}L = \frac{d}{dt}(mv) = m\dot{v}$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{q}}L'\right) - \frac{\partial}{\partial q}L' = \frac{d}{dt}(mv - u) = m\dot{v}$$

with the obvious extension to more than one dimension.

8. Non-commuting derivatives: Use the definition of the total time derivativet to

a) show that

$$\frac{\partial}{\partial q}\dot{f} = \frac{d}{dt}\frac{\partial}{\partial q}f$$

i.e., these derivatives commute for any function $f = f(q, \dot{q}, t)$.

Solutions:

Expand the total derivative in terms of partials

$$\begin{aligned}\frac{\partial}{\partial q}\frac{d}{dt}f &= \frac{\partial}{\partial q}\left(\frac{\partial}{\partial t}f + \dot{q}\frac{\partial}{\partial q}f + \ddot{q}\frac{\partial}{\partial\dot{q}}f\right)\\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial q}f + \left(\frac{\partial}{\partial q}\dot{q}\right)\frac{\partial}{\partial q}f + \dot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial q}f + \left(\frac{\partial}{\partial q}\ddot{q}\right)\frac{\partial}{\partial\dot{q}}f + \ddot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial\dot{q}}f\end{aligned}$$

and similarly

$$\frac{d}{dt}\frac{\partial}{\partial q}f = \frac{\partial}{\partial t}\frac{\partial}{\partial q}f + \dot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial q}f + \ddot{q}\frac{\partial}{\partial\dot{q}}\frac{\partial}{\partial q}f$$

Now, we see that

$$\frac{\partial}{\partial q}\frac{d}{dt}f - \frac{d}{dt}\frac{\partial}{\partial q}f = \left(\frac{\partial}{\partial q}\dot{q}\right)\frac{\partial}{\partial q}f + \left(\frac{\partial}{\partial q}\ddot{q}\right)\frac{\partial}{\partial\dot{q}}f$$

but remember, $q = q(t)$ so that $\frac{d}{dt}q = \frac{\partial}{\partial t}q$. This means

$$\frac{\partial}{\partial q}\dot{q} = \frac{\partial}{\partial q}\frac{\partial}{\partial t}q = \frac{\partial}{\partial t}\frac{\partial}{\partial q}q = \frac{\partial}{\partial t}1 = 0$$

and similarly for $\frac{\partial}{\partial q}\ddot{q}$. Thus, we see that

$$\frac{\partial}{\partial q}\frac{d}{dt}f - \frac{d}{dt}\frac{\partial}{\partial q}f = 0$$

b) show that

$$\frac{\partial}{\partial\dot{q}}\dot{f} = \frac{d}{dt}\frac{\partial}{\partial\dot{q}}f + \frac{\partial}{\partial q}f$$

(i.e., these derivatives do NOT commute.)

Solutions:

Again, expand the total derivative in terms of partials

$$\begin{aligned}\frac{\partial}{\partial\dot{q}}\frac{d}{dt}f &= \frac{\partial}{\partial\dot{q}}\left(\frac{\partial}{\partial t}f + \dot{q}\frac{\partial}{\partial q}f + \ddot{q}\frac{\partial}{\partial\dot{q}}f\right)\\ &= \frac{\partial}{\partial t}\frac{\partial}{\partial\dot{q}}f + \frac{\partial}{\partial q}f + \dot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial\dot{q}}f + \left(\frac{\partial}{\partial\dot{q}}\ddot{q}\right)\frac{\partial}{\partial\dot{q}}f + \ddot{q}\frac{\partial}{\partial\dot{q}}\frac{\partial}{\partial\dot{q}}f\end{aligned}$$

and

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}f = \frac{\partial}{\partial t}\frac{\partial}{\partial \dot{q}}f + \dot{q}\frac{\partial}{\partial q}\frac{\partial}{\partial \dot{q}}f + \ddot{q}\frac{\partial}{\partial \dot{q}}\frac{\partial}{\partial \dot{q}}f$$

which shows that

$$\frac{\partial}{\partial \dot{q}}\frac{d}{dt}f - \frac{d}{dt}\frac{\partial}{\partial \dot{q}}f = \frac{\partial}{\partial q}f + \left(\frac{\partial}{\partial \dot{q}}\ddot{q}\right)\frac{\partial}{\partial \dot{q}}f$$

Once again, we recognize that $\dot{q} = \dot{q}(t)$ so that $\ddot{q} = \frac{\partial}{\partial t}\dot{q}$. This implies

$$\frac{\partial}{\partial \dot{q}}\ddot{q} = \frac{\partial}{\partial \dot{q}}\frac{\partial}{\partial t}\dot{q} = \frac{\partial}{\partial t}\frac{\partial}{\partial \dot{q}}\dot{q} = \frac{\partial}{\partial t}1 = 0$$

and we are left with

$$\frac{\partial}{\partial \dot{q}}\frac{d}{dt}f - \frac{d}{dt}\frac{\partial}{\partial \dot{q}}f = \frac{\partial}{\partial q}f$$

Because $\frac{\partial}{\partial q}f$ does not vanish in general, we see that the operators $\frac{\partial}{\partial \dot{q}}$ and $\frac{d}{dt}$ do not commute.

9. Take $L = \frac{1}{2}mv^2 - mgz$

a) Find the equations of motion.

Solutions:

From the Lagrangian,

$$L = \frac{1}{2}mv^2 - mgz$$

We obtain

$$\frac{\partial L}{\partial z} = -mg$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

By using the Lagrange's equations

$$m\ddot{z} = -mg \Rightarrow \ddot{z} = -g$$

In the same way

$$\ddot{x} = \ddot{y} = 0$$

b) Take $\vec{x}(0) = 0, \vec{v}(0) = \vec{v}_0, v_{0z} > 0$ and find $\vec{x}(\tau)$ and $\vec{v}(\tau)$, τ such that $z(\tau) = 0, \tau \neq 0$

Solutions:

From the equations of motion, we obtain

$$\vec{x}(t) = v_{0x}t\hat{x} + v_{0y}t\hat{y} + v_{0z}t\hat{z} - \frac{1}{2}gt^2$$

$$z(\tau) = 0 \ (\tau \neq 0) \Rightarrow \tau = \frac{2v_{0z}}{g}$$

Thus,

$$\vec{x}(\tau) = \frac{2v_{0x}v_{0z}}{g}\hat{x} + \frac{2v_{0y}v_{0z}}{g}\hat{y}$$

$$\vec{v}(\tau) = v_{0x}\hat{x} + v_{0y}\hat{y} - v_{0z}\hat{z}$$

10. A particle of mass m is confided to a parabolic surface of rotation $z = a\rho^2$, where $\rho = \sqrt{x^2 + y^2}$. The gravitational potential is $U = mgz$.

a) Show the Lagrangian is

$$L = \frac{1}{2}m(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\phi}^2) - mgz$$

subject ot the constraint that $z = a\rho^2$, i.e. the particle remains on the surface of parbolic rotation.

Solutions:

The Lagrangian is $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$ and if we define the angle ϕ so that

$$x = \rho \sin\phi$$

$$y = \rho \cos\phi$$

we then recognize that

$$\dot{x}^2 + \dot{y}^2 = \left(\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi}\right)^2 + \left(\dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi}\right)^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2$$

which simplifies the Lagrangian to

$$L = \frac{1}{2}m(\dot{z}^2 + \dot{\rho}^2 + \rho^2\dot{\phi}^2) - mgz$$

b) Use the constraint equation to eliminate z from the Lagrangian and give the equations of motion.

Solutions:

$$z = a\rho^2 \Rightarrow \dot{z} = 2a\rho\dot{\rho}$$

so that

$$L = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + 4a^2\rho^2\dot{\rho}^2\right) - mga\rho^2$$

Now, using the Euler-Lagrange equations yields

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\phi}}L\right) - \frac{d}{d\phi}L = \frac{d}{dt}\left(m\rho^2\dot{\phi}\right) = 0$$

so that

$$2\dot{\rho}\dot{\phi} + \rho\ddot{\phi} = 0$$

and

$$\frac{d}{dt}\left(\frac{\partial}{\partial\dot{\rho}}L\right) - \frac{d}{d\rho}L = \frac{d}{dt}\left(m\dot{\rho} + 4ma^2\rho^2\dot{\rho}\right) - \left(m\dot{\phi}^2\rho + 4ma^2\rho\dot{\rho}^2 - 2mga\rho\right) = 0$$

so that

$$\left(1 + 4a^2\rho\right)\ddot{\rho} + 4a^2\rho\dot{\rho}^2 - \dot{\phi}^2\rho + 2ga\rho = 0$$

c) Find all momenta from the Lagrangian. Which are conserved and why?

Solutions:

$$p_\rho = \frac{\partial}{\partial\dot{\rho}}L = m\dot{\rho} + 4ma^2\rho^2\dot{\rho}$$

$$p_\phi = \frac{\partial}{\partial\dot{\phi}}L = m\rho^2\dot{\phi}$$

p_ϕ is conserved ($\dot{p}_\phi = 0$) because $\frac{\partial}{\partial \phi} L = 0$ and p_ρ is *not* conserved ($\dot{p}_\rho \neq 0$) because $\frac{\partial}{\partial \rho} L \neq 0$.

d) OPTIONAL: Solve the equations of motion. Don't spend too much time on this!

Solutions:

p_ϕ is conserved, so we can solve for $\phi(t)$ given $\rho(t)$.

$$\phi(t) = \phi_0 + \int_{t_0}^{t} d\tau \frac{p_\phi}{m\rho(\tau)^2}$$

We can also reduce this set of equations to a single (non-linear) ordinary differential equation for ρ

$$\left(1 + 4a^2\rho\right)\ddot{\rho} + 4a^2\rho\dot{\rho}^2 - \left(\frac{p_\phi}{m\rho^2}\right)^2 \rho + 2ga\rho = 0$$

This is a very hard equation to solve, but we can at least solve for $\rho(t)$ implicitly. Applying a Legendre transformation, we obtain the Hamiltonian (H)

$$H = p_\phi\dot{\phi} + p_\rho\dot{\rho} - L = \frac{1}{2}m\left(\dot{\rho}^2 + \frac{p_\phi^2}{m^2\rho^2} + 4a^2\rho^2\dot{\rho}^2\right) + mga\rho^2$$

We then note that H is conserved because $\dot{H} = \frac{\partial}{\partial t} H = 0$. Thus, we can obtain

$$\frac{E}{m} = \epsilon = \frac{1}{2}\left(\dot{\rho}^2 + \frac{p_\phi^2}{m^2\rho^2} + 4a^2\rho^2\dot{\rho}^2\right) + ga\rho^2 \Rightarrow \dot{\rho}^2 = \frac{2\epsilon - ga\rho^2 - \frac{p_\phi^2}{m^2\rho^2}}{1 + 4a\rho^2}$$

Based on the initial conditions, we can then choose a sign for $\dot{\rho}$ and integrate forward. Let's assume $\dot{\rho} > 0$ for now. Then we have

$$t = \int_{\rho_0}^{\rho} dr \sqrt{\frac{1 + 4ar^2}{2\epsilon - 2gar^2 - p_\phi^2/m^2r^2}}$$

which will hold as long as $\dot{\rho} > 0$. At the zero crossings, we have to determine whether the sign of $\dot{\rho}$ changes and continue the integration piece-wise if necessary. There is a chance that a closed form solution to this integral exists, but it isn't likely to be a friendly function.

However, we can say a few more quantitative things about this system. Let's look for points where the radial velocity vanishes ($\dot{\rho} = 0$).

$$\epsilon = ga\rho^2 + \frac{(p_\phi/m)^2}{2\rho^2}$$

which allows us to solve for the radial bounds on the orbit

$$\frac{\epsilon - \sqrt{\epsilon^2 - 2ga(p_\phi/m)^2}}{2ga} \leq \rho^2 \leq \frac{\epsilon + \sqrt{\epsilon^2 - 2ga(p_\phi/m)^2}}{2ga}$$

and we conclude that the particle executes some sort of complicated periodic motion in ρ, "bouncing" between the inner and outer boundaries while spinning around the $\hat{z}$ axis according to $\dot{\phi} = p_\phi/m\rho^2$. Integrating between $\rho_{\min}$ and $\rho_{\max}$ will give the period of radial oscillation.

$$T = 2\int_{\rho_{\min}}^{\rho_{\max}} dr \sqrt{\frac{1 + 4ar^2}{2\epsilon - 2gar^2 - p_\phi^2/m^2r^2}}$$

We also note that this gives us the conditions for circular orbits

$$\epsilon^2 = 2ga(p_\phi/m)^2$$

$$\rho_{\text{circ}} = \sqrt{\frac{\epsilon}{2ga}}$$

$$\dot{\phi} = \frac{p_\phi}{m\rho_{\text{circ}}^2} = \sqrt{2ga} = \text{constant}$$

And integrating between ρ_{min} and ρ_{max} will give the period of radial oscillation.

If we consider small perturbations around circular orbits ($\rho = \rho_{\text{circ}} + r \; | r \ll \rho$), we can show that the energy is

$$\epsilon - \epsilon_{\text{circ}} = \frac{1}{2}(1 + 4a^2\rho_{\text{circ}}^2)\dot{r}^2 + gar^2$$

which we recognize as the Hamiltonian for a simple harmonic oscillator with period

$$T = 2\pi\sqrt{\frac{1 + 4a\rho_{\text{circ}}^2}{2ga}}$$

Taking the ratio of this period with the orbital period yields

$$\frac{T}{2\pi/\dot{\phi}_{\text{circ}}} = \sqrt{1 + 4a\rho_{\text{circ}}^2}$$

If this is a rational number, then the small orbits will close. Notice that this depends on the radius of the circular orbit, which suggest that the general orbits, in general, do not close.

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