Massachusetts Institute of Technology

8.223, Classical Mechanics II IAP 2017

Solutions 2

11. For a particle described by $L = \frac{1}{2}mv^2 - U$,

a) Show that if $U = U(\vec{x}), E = T + U$ is constant.

Solutions:

$$L = \frac{1}{2}mv^2 - U(\vec{x})$$
$$E \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

From the Lagrangian,

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i \Rightarrow \sum_i \frac{\partial L}{\partial \dot{x}_i}\dot{x}_i = m\dot{x}\dot{x} + m\dot{y}\dot{y} + m\dot{z}\dot{z} = mv^2$$

$$\Rightarrow E = mv^2 - (\frac{1}{2}mv^2 - U(\vec{x})) = \frac{1}{2}mv^2 + U(\vec{x}) = T + U$$

From the lecture, we know that

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial L}{\partial t} = \frac{d}{dt}\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial t}$$

So,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L\right) = -\frac{\partial L}{\partial t}$$

Thus, if L doesn't depend explicitly on time, i.e., $\frac{\partial L}{\partial t} = 0$,

$$\frac{dE}{dt} = 0$$

E is constant.

b) Show that if $U = U(\vec{x}, \dot{\vec{x}})$, the E is conserved, but not necessarily $E = T + U$.

Solutions:

Part (a) showed that E is conserved, since $\dfrac{\partial L}{\partial t} = 0$, however

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \left(2T - \sum_i \frac{\partial U}{\partial \dot{q}_i}\dot{q}_i\right) - (T - U) = T + U - \sum_i \frac{\partial U}{\partial \dot{q}_i}\dot{q}_i$$

12. For $L = T_a + T_b - U(\vec{x}_a - \vec{x}_b)$, show that $\vec{P} = \vec{p}_a + \vec{p}_b$ is conserved. Hint: try defining new coordinates:

$$\begin{aligned}\vec{X} &= \vec{x}_a + \vec{x}_b \\ \vec{x} &= \vec{x}_a - \vec{x}_b.\end{aligned}$$

Solutions:

I'll present two ways of looking at this. Calculate $\dot{\vec{p}}_a$ and $\dot{\vec{p}}_b$. First, a note on notation: $\dfrac{\partial f}{\partial \vec{x}} = \sum_i \dfrac{\partial f}{\partial x_i}\hat{x}_i$, where x_i is a coordinate in your coordinate system (so x, y and z in Cartesian).

$$\begin{aligned}\dot{\vec{p}}_a &= \frac{\partial L}{\partial \vec{x}_a} = \frac{\partial L}{\partial(\vec{x}_a - \vec{x}_b)}\frac{\partial(\vec{x}_a - \vec{x}_b)}{\partial \vec{x}_a} = \frac{\partial U}{\partial(\vec{x}_a - \vec{x}_b)} \\ \dot{\vec{p}}_b &= \frac{\partial L}{\partial \vec{x}_b} = \frac{\partial L}{\partial(\vec{x}_a - \vec{x}_b)}\frac{\partial(\vec{x}_a - \vec{x}_b)}{\partial \vec{x}_b} = -\frac{\partial U}{\partial(\vec{x}_a - \vec{x}_b)}\end{aligned}$$

Then it's clear that $\dfrac{d}{dt}(\vec{p}_a + \vec{p}_b) = 0$, hence the $\vec{p}_a + \vec{p}_b$ is conserved.

Another option is to change variables to $\vec{x} = \vec{x}_a - \vec{x}_b$ and $\vec{X} = \vec{x}_a + \vec{x}_b$. Solving this gives $\vec{x}_a = \frac{1}{2}(\vec{x} + \vec{X})$ and $\vec{x}_b = \frac{1}{2}(\vec{X} - \vec{x})$. Substituting this in to the Lagrangian gives

$$L = \frac{1}{8}m_a(\dot{\vec{X}} + \dot{\vec{x}})^2 + \frac{1}{8}m_b(\dot{\vec{X}} - \dot{\vec{x}})^2 - U(\vec{x})$$

Then

$$\frac{\partial L}{\partial \vec{X}} = 0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{X}}}\right)$$

$$\frac{\partial L}{\partial \dot{\vec{X}}} = \frac{1}{4}m_a(\dot{\vec{X}} + \dot{\vec{x}}) + \frac{1}{4}m_b(\dot{\vec{X}} - \dot{\vec{x}}) = \frac{1}{2}(m_a\dot{\vec{x}}_a + m_b\dot{\vec{x}}_b) = \frac{1}{2}\vec{p}$$

which implies

$$\frac{d\vec{p}}{dt} = 0$$

hence $\vec{p}$ is conserved.

13. For a two body system with particle 1 of mass m_1 located at $\vec{x}_1$ and particle 2 of mass m_2 located at $\vec{x}_2$, find the center of mass $\vec{R}$ in terms of m_1, m_2, $\vec{x}_1$ and $\vec{x}_2$.

Solutions:

$$\left(\sum_i m_i\right)\vec{R} = \sum_i m_i\vec{x}_i \Rightarrow \vec{R} = \frac{\sum_i m_i\vec{x}_i}{\sum_i m_i} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$

14. a) Show that the change in coordinates of a particle (fixed position) from a small rotation by ϕ around the $\hat{z}$ axis may be written as

$$d\vec{x} = \vec{x}' - \vec{x} = (\phi\hat{z}) \times \vec{x}.$$

Solutions:

A simple way to deal with a rotation about the z-axis is to look at it in polar coordinates. You start at a position (x, y) and rotate anti-clockwise around z-axis. Converting back to Cartesian gives

$$x' = x\cos\phi - y\sin\phi$$

$$y' = y\cos\phi + x\sin\phi$$

$$z' = z$$

In the small angle limit ($\phi \ll 1$), $\sin(\phi) = \phi$ and $\cos(\phi) = 1$, giving $x' = x + y\phi$ and $y' = y - x\phi$, giving

$$\begin{aligned} dx &= x' - x = -y\phi \\ dy &= y' - y = +x\phi \\ dz &= z' - z = 0 \end{aligned}$$

This is exactly $d\vec{x} = (\phi\hat{z}) \times \vec{x}$.

b) OPTIONAL: Show that the change in coordinates $d\vec{x}$ for a small rotation around the axis $\vec{b}$ by ϕ may be written as $d\vec{x} = (\phi\hat{b}) \times \vec{x}$.

Solutions:

There was nothing special about $\hat{z}$ in part (a), so the same must be true for any direction.

15. Moving pendulum: A plane pendulum consists of a bob of mass m suspended by a massless rigid rod of length l that is hinged to a sled of mass M. The sled slides without friction on a horizontal rail (see Fig. 1).

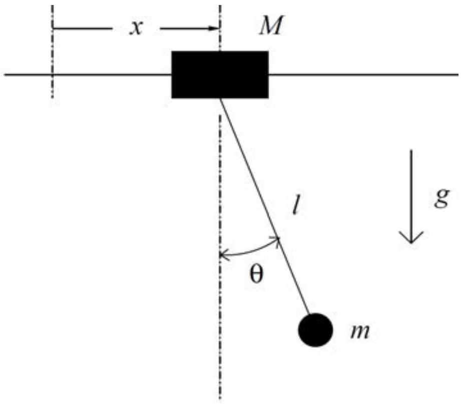

Figure 1: moving pendulum

a) ×2) Write the Lagrangian for the system and derive the equations of motion.

Solutions:

We start by computing the Cartesian coordinates of the bob (setting the y position of the sled is 0)

$$x_1 = x + l\sin\theta \Rightarrow \dot{x}_1 = \dot{x} + \dot{\theta}l\cos\theta$$

$$y_1 = -l\cos\theta \Rightarrow \dot{y}_1 = \dot{\theta}l\sin\theta$$

thus

$$T = \frac{m}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{M}{2}\dot{x}^2 = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}(2\dot{\theta}\dot{x}l\cos\theta + \dot{\theta}^2 l^2)$$

and

$$V = -mgl\cos\theta$$

hence

$$L = T - V = \frac{M+m}{2}\dot{x}^2 + \frac{m}{2}(2\dot{\theta}\dot{x}l\cos\theta + \dot{\theta}^2 l^2) + mgl\cos\theta$$

Then we use Lagrange's equations to derive the equations of motion for the system. For x we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - m\dot{\theta}^2 l\sin\theta = 0$$

and for θ

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\ddot{x}l\cos\theta + ml^2\ddot{\theta} + mgl\sin\theta = 0$$

b) At time $t = 0$ the bob and the sled, which had previously been at rest (with $\theta = 0$), are set in motion by a sharp tap delivered to the bob. The tap imparts a horizontal impulse $\Delta P = F\Delta t$ to the bob. Find expressions for the values of $\dot{\theta}$ and $\dot{x}$ just after the impulse. [Hint: consider both linear and angular momentum.]

Solutions:

Applying the impulse-momentum theorem, for linear momentum we get

$$\Delta P = \sum_n \Delta p_n = m(\dot{x} + \dot{\theta}l) + M\dot{x}$$

and for angular momentum, choosing the hanging bob as the momentary origin for computing the angular momentum (since impulse is imparted there, the change in angular momentum about that point is zero)

$$\Delta L = 0 = \sum_n L_n = Ml\dot{x} \Rightarrow \dot{x} = 0$$

Combining the above two equations, we get

$$\dot{\theta} = \frac{\Delta P}{ml}$$

16. (×2) Flyball Governor: In the system shown in Fig. 2, the particle m_2 moves on the vertical axis and the whole system rotates about this axis with a constant angular velocity Ω. Derive the Lagrangian of the system and obtain the equations of motion.

Solutions:

Let's compute the Cartesian coordinates of one of the masses with $m = m_1$ first,

$$x_1 = a\sin\theta\cos(\Omega t)$$

$$y_1 = a\sin\theta\sin(\Omega t)$$

$$z_1 = -a\cos\theta$$

Accordingly,

$$\dot{x}_1 = a\dot{\theta}\cos\theta\cos(\Omega t) - a\Omega\sin\theta\sin(\Omega t)$$

$$\dot{y}_1 = a\dot{\theta}\cos\theta\sin(\Omega t) + a\Omega\sin\theta\cos(\Omega t)$$

$$\dot{z}_1 = a\dot{\theta}\sin\theta$$

thus,

$$T_1 = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2 + \dot{z}_1^2) = \frac{m_1}{2}a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta)$$

$$V_1 = -m_1 g a\cos\theta$$

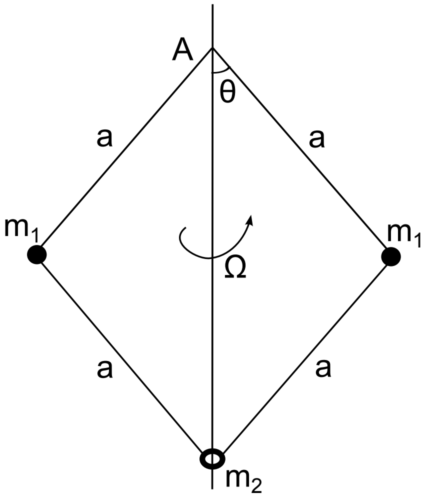

Figure 2: Centrifugal governor (aka, flyball governor)

For m_2,

$$x_2 = y_2 = 0$$

$$z_2 = -2a\cos\theta$$

then

$$\dot{x}_2 = \dot{y}_2 = 0$$

$$\dot{z}_2 = 2a\dot{\theta}\sin\theta$$

so,

$$T_2 = 2m_2a^2\dot{\theta}^2\sin^2\theta$$

$$V_2 = -2m_2ga\cos\theta$$

The total Lagrangian is

$$L = 2(T_1 - V_1) + (T_2 - V_2) = m_1a^2(\dot{\theta}^2 + \Omega^2\sin^2\theta) + 2m_2a^2\dot{\theta}^2\sin^2\theta + 2(m_1 + m_2)ga\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = 2m_1a^2\dot{\theta} + 4m_2a^2\dot{\theta}\sin^2\theta$$

$$\frac{\partial L}{\partial\theta} = (2m_1a^2\Omega^2 + 4m_2a^2\dot{\theta}^2)\sin\theta\cos\theta - 2(m_1 + m_2)ga\sin\theta$$

The equation of motion is

$$m_1a^2(\ddot{\theta} - \Omega^2\sin\theta\cos\theta) + 2m_2a^2(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\sin\theta\cos\theta) + (m_1 + m_2)ga\sin\theta = 0$$

17. (×2) Least time path: A particle can slide (without any friction) under gravity from point A$(0, 0, h)$ to point B$(X, 0, 0)$ along a curve $z(x)$ (Fig. 3). Find the path $z(x)$ that minimizes the time. The particle starts at rest from point A. [Hint: you can use conservation laws to simplify your calculation. You may encounter integrals of the form

$$\int\sqrt{\frac{a-t}{b+t}}\,\mathrm{d}t$$

which can be solved by substituting $t = a - (a + b)\sin^2\theta$.]

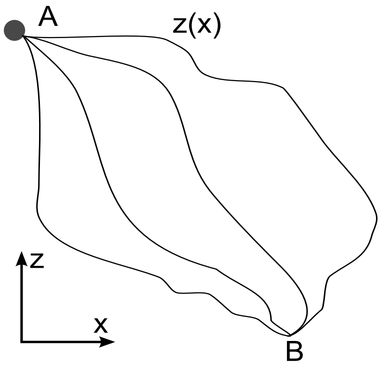

Figure 3: Least time path

Solutions:

By conservation of energy, the particle moves with speed $v^2(z) = 2g(h-z)$. By inspection, we also know that there is no motion in the $\hat{y}$ direction. We then have that

$$v^2 = \dot{x}^2 + \dot{z}^2 = \left(\frac{dx}{dt}\right)^2 [1 + z'^2] = 2g(h-z)$$

where $z' = dz/dx$. Taking the square root of both sides and rearranging,

$$dt = \left[\frac{1+z'^2}{2g(h-z)}\right]^{1/2} dx$$

so that the time taken to travel from $x = 0$ to $x = X$ along path $z(x)$ is given by $\tau[z(x)] = \int_0^X L(z, z', x)dx$ with

$$L(z, z', x) = \left[\frac{1+z'^2}{2g(h-z)}\right]^{1/2}$$

Since L doesn't depend explicitly on x, the path $z(x)$ that extremizes $\tau[z(x)]$ satisfies the following relation (i.e., the Hamiltonian is conserved)

$$\text{constant} = z'\frac{\partial L}{\partial z'} - L = -\frac{L}{1+z'^2} = -\left[\frac{1}{2g(h-z)(1+z'^2)}\right]^{1/2}$$

Rearranging and integrating,

$$\int_h^{z(x)} \left(\frac{h-z}{a+z}\right) dz = x$$

where a is the constant and we have used the boundary condition that $z(x=0) = h$. Substituting $z = h - (a+h)\sin^2\theta$ and integrating,

$$x = (a+h)[\theta - \frac{1}{2}\sin 2\theta]$$

Writing $\phi = 2\theta$, this becomes

$$x = \frac{a+h}{2}[\phi - \sin\phi]$$

$$z = h - \frac{a+h}{2}[1 - \cos\phi]$$

which is a cycloid with $(x, z) = (0, h)$ at its cusp. The constant a is determined by the condition that the curve passes through the endpoint $(X, 0)$.

18. Show that

$$\frac{d}{dt}(\vec{x} \cdot \dot{\vec{x}}) = v^2 + \vec{x} \cdot \ddot{\vec{x}}.$$

Solutions:

$$\frac{d}{dt}(\vec{x} \cdot \dot{\vec{x}}) = \dot{\vec{x}} \cdot \dot{\vec{x}} + \vec{x} \cdot \ddot{\vec{x}} = v^2 + \vec{x} \cdot \ddot{\vec{x}}$$

19. Show that a potential $U = \kappa r^k$ results in a force $\vec{F} = -k\kappa r^{k-1}\hat{r}$.

Solutions:

First, note that

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$$

same for y and z. Then for $U = \kappa r^k$

$$\frac{\partial U}{\partial x} = \frac{dU}{dr}\frac{\partial r}{\partial x} = \kappa k r^{k-1}\frac{x}{r} = \kappa k r^{k-2} x$$

So

$$\vec{F} = -\vec{\nabla}U = -\kappa k r^{k-2}(x\hat{x} + y\hat{y} + z\hat{z}) = -\kappa k r^{k-1}\frac{x\hat{x} + y\hat{y} + z\hat{z}}{r} = -\kappa k r^{k-1}\hat{r}$$

Alternately, note that in spherical coordinates

$$\vec{\nabla}U = \frac{\partial U}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \phi}\hat{\phi}$$

from which the answer follows trivially since $\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \phi} = 0$.

20. Carry out integration of Eq. 14.7 in Landau for $U = -\alpha/r$. Use the substitution $u = 1/r$ in Eq. 14.7 and show $u = (\epsilon/p)\cos\phi + (1/p)$ solves the integral. Find p and ϵ in terms of L, μ, α and E. Hint: multiply the numerator and denominator by r^2.

Solutions:

Start with Landau's equation and make a substitution $u = r^{-1}$ so $du = r^{-2}dr$

$$\phi(r) = \int \frac{Lr^{-2}dr}{\sqrt{2\mu(E + \alpha r^{-1}) - L^2 r^{-2}}} = \int \frac{-Ldu}{\sqrt{2\mu(E + \alpha u) - L^2 u^2}}$$

Complete the square in the denominator:

$$\frac{\mu^2\alpha^2}{L^2} - (Lu - \frac{\mu\alpha}{L})^2 = -Lu^2 + 2\mu\alpha u - \frac{\mu^2\alpha^2}{L^2} + \frac{\mu^2\alpha^2}{L^2}$$

putting this in the integral gives

$$\int \frac{-Ldu}{\sqrt{2\mu E + \frac{m^2\alpha^2}{L^2} - (Lu - \frac{\mu\alpha}{L})^2}} = -\frac{L}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}\int \left(1 - \left(\frac{Lu - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}\right)^2\right)^{-1/2} du$$

Now make another substitution $y = \dfrac{Lu - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}$, giving $dy = \dfrac{L}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} du$. Now we have

$$\frac{L}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} \frac{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}{L} \int \frac{-dy}{\sqrt{1-y^2}} = \cos^{-1}(y) + C$$

Substituting in for y and dropping the integration constant C (which is just an arbitrary phase) we get

$$\phi(r) = \cos^{-1}\left(\frac{Lr^{-1} - \mu\alpha L^{-1}}{\sqrt{2\mu E + \mu^2\alpha^2 L^{-2}}}\right)$$

Which we rewrite as

$$\sqrt{2\mu E + \frac{\mu^2\alpha^2}{M^2}} \cos(\phi) + \frac{\mu\alpha}{L} = \frac{L}{r}$$

Matching terms to $\frac{p}{r} = \epsilon\cos(\phi) + 1$ gives

$$p = \frac{L^2}{\mu\alpha}$$

and

$$\epsilon = \frac{L}{\mu\alpha}\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}} = \sqrt{\frac{2L^2E}{\mu\alpha^2} + 1}$$

21. Use $p/r = \epsilon\cos\phi + 1$ to show $E > 0$ corresponds to a hyperbola, $E = 0$ corresponds to a parabola, and $E < 0$ corresponds to an ellipse.

    Solutions:

    It's simplest to show this by noting that for a conic section, if $\epsilon = 1$ you get a parabola, if $0 < \epsilon < 1$ you get an ellipse and if $\epsilon > 1$ you get a hyperbola. Since we have a formula for ϵ, which we worked hard to get in the previous exercise, we just see what energies give us what we want.

    Putting in $E = 0$ gives $\epsilon = 1$, a parabola

    Putting in $E < 0$ gives $\epsilon = \sqrt{1 - \frac{2L^2|E|}{m\alpha^2}} < 1$, an ellipse. $\epsilon < 1$ since you have $\sqrt{1-x}$, where x is positive

    Putting in $E > 0$ gives $\epsilon = \sqrt{1 + \frac{2L^2|E|}{m\alpha^2}} > 1$, a hyperbola

22. There are plenty of asteroids flying around in the universe. For an asteroid coming from infinity with velocity v_∞, find the effective total cross-section for the asteroid to hit the Earth (due to the gravitational attraction between them). You can assume that the asteroid is much smaller than the earth. What happens in the limit $v_\infty = \infty$? Interpret the result.

    Hint: Treat this as a scattering problem with the asteroids all moving in the same direction (as in a beam of particles), and compute the maximum impact factor for an asteroid to hit the Earth, $b_{\max}$. The cross-section is then $\sigma = \pi b_{\max}^2$.

    Solutions:

    The condition for a particle (asteroid, mass m) to hit the Earth is that $r_{\min} < R_\oplus$, where $r_{\min}$ is the point on the path which is nearest to the center of the sphere. The greatest possible value of b is given by

$$r_{\min} = R$$

The Hamiltonian is conserved for this system, so

$$\text{constant} = H = \frac{1}{2}\mu\dot{r}^2 - \frac{GM\mu}{r} + \frac{L^2}{2\mu r^2} = \frac{1}{2}\mu v_\infty^2$$

and we know that the angular momentum is conserved, so

$$L = \mu b v_\infty$$

At the point of closest approach, $\dot{r} = 0$ and we obtain

$$-\frac{GM\mu}{R_\oplus} + \frac{L^2}{2\mu R_\oplus^2} = \frac{1}{2}\mu v_\infty^2$$

so that

$$2\mu R_\oplus^2\left(\frac{1}{2}\mu v_\infty^2 + \frac{GM\mu}{R_\oplus}\right) = L^2 = (\mu b v_\infty)^2$$

which yields

$$b^2 = R_\oplus^2\left(1 + \frac{2GM/v_\infty^2}{R_\oplus}\right)$$

and we finally obtain

$$\sigma = \pi b^2 = \pi R_\oplus^2\left(1 + \frac{2GM/v_\infty^2}{R_\oplus}\right) \geq \pi R_\oplus^2$$

8.223 Classical Mechanics II
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