Massachusetts Institute of Technology

8.223, Classical Mechanics II — IAP 2017

Solutions 3

23. Verify the Virial Theorem for a one dimensional simple harmonic oscillator by direct calculation, i.e. compute $T(t)$ and $U(t)$ and find their averages over one cycle.

Solutions:

For a 1-D simple harmonic oscillator, the general solution is $x(t) = A\cos(\omega t + \phi_0)$, giving $\dot{x}(t) = -\omega A \sin(\omega t + \phi_0)$, where $\omega^2 = \frac{k}{m}$. The Virial theorem states $2\bar{T} = p\bar{U}$. For a simple harmonic oscillator, $U = \frac{1}{2}kx^2$, so $p = 2$ and $\bar{T} = \bar{U}$. Average over one full period $T = \dfrac{2\pi}{\omega}$

$$\bar{T} = \frac{1}{T}\int_0^T \frac{1}{2}m\dot{x}^2 dt = \frac{m\omega^2}{2}A^2\frac{1}{T}\int_0^T \sin^2(\omega t + \phi_0)dt = \frac{mk}{2m}A^2\frac{1}{2} = \frac{kA^2}{4}$$

$$\bar{U} = \frac{1}{T}\int_0^T \frac{1}{2}kx^2 dt = \frac{k}{2}A^2\frac{1}{T}\int_0^T \cos^2(\omega t + \phi_0)dt = \frac{k}{2}A^2\frac{1}{2} = \frac{kA^2}{4}$$

Where I have used the identity $\displaystyle\int_0^{2\pi} \sin^2(x+a)dx = \pi$, for any constant a. Note that $\sin(x + \frac{\pi}{2} + a) = \cos(x + a)$.

24. Compute the cross-section for back-scattering off a fixed impenetrable sphere of radius R (i.e., $U = 0$ for $r > R$, and $U = \infty$ for $r \leq R$, and scattering angle $|\theta| > \pi/2$).

Solutions:

To scatter into $\theta > \pi/2$, the incoming particle must hit the sphere with $b < R\sin(\pi/4) = R/\sqrt{2}$. Thus, $\sigma = \pi R^2/2$.

25. Show that a solution to

$$\ddot{x} + \omega_o^2 x = \frac{F}{m}\cos(\omega t + \theta)$$

for the case of resonant driving ($\omega_o = \omega$) is $x(t) = a_1\cos(\omega T + \phi) + a_2 t\sin(\omega_o t + \theta)$. Find the constants a_1 and ϕ for the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_o$.

Solutions:

We attack this via direct substitution

$$x = a_1\cos(\omega_o t + \phi) + a_2 t\sin(\omega_o t + \theta)$$

$$\dot{x} = -a_1\omega_o\sin(\omega_o t + \phi) + a_2 t\omega_o\cos(\omega_o t + \theta) + a_2\sin(\omega_o t + \theta)$$

$$\ddot{x} = -a_1\omega_o^2\cos(\omega_o t + \phi) - a_2 t\omega_o^2\sin(\omega_o t + \theta) + 2a_2\omega_o\cos(\omega_o t + \theta) = -\omega_o^2 x + 2a_2\omega_o\cos(\omega_o t + \theta)$$

so that

$$\ddot{x} + \omega_o^2 x = 2a_2\omega_o \cos(\omega_o t + \theta) = \frac{F}{m}\cos(\omega_o t + \theta) \Rightarrow a_2 = \frac{F}{2m\omega_o}$$

Now, we enforce the initial conditions

$$x(0) = 0 = a_1 \cos\phi$$

$$\dot{x}(0) = v_o = -a_1\omega_o \sin\phi + a_2 \sin\theta = \frac{F}{2m\omega_o}\sin\theta - \omega_o a_1 \sin\phi$$

From the initial velocity, we know that $a \neq 0$ and we chose

$$\phi = \pi/2 \Rightarrow \cos\phi = 0$$

This yields

$$a_1 = \frac{F\sin\theta}{2m\omega_o^2} - \frac{v_o}{\omega_o}$$

26. (×2) Small Oscillations: For the system in problem 16 (pset 2), compute the angular frequency ω for small oscillations about (stable) equilibrium.

Solutions:

Flyball governor: The equation of motion is

$$m_1 a^2\left(\ddot{\theta} - \Omega^2 \sin\theta\cos\theta\right) + 2m_2 a^2\left(\ddot{\theta}\sin^2\theta + \dot{\theta}^2\sin\theta\cos\theta\right) + (m_1+m_2)ga\sin\theta = 0$$

and the only fixed point occurs at

$$-m_1 a^2\Omega^2 \sin\theta_o\cos\theta_o + (m_1+m_2)ga\sin\theta_o = 0$$

$$\cos\theta_o = \left(\frac{m_1+m_2}{m_1}\right)\frac{g}{a\Omega^2}$$

$$\sin\theta_o = \sqrt{1-\cos^2\theta_o}$$

We now expand around the fixed point: $\theta = \theta_o + \epsilon$

$$\begin{aligned} &m_1 a^2(\ddot{\epsilon} - \Omega^2(\sin\theta_o\cos\epsilon + \cos\theta_o\sin\epsilon)(\cos\theta_o\cos\epsilon - \sin\theta_o\sin\epsilon) \\ &\quad + 2m_2a^2(\ddot{\epsilon}(\sin\theta_o\cos\epsilon + \cos\theta_o\sin\epsilon)^2 + \dot{\epsilon}^2(\sin\theta_o\cos\epsilon + \cos\theta_o\sin\epsilon)(\cos\theta_o\cos\epsilon - \sin\theta_o\sin\epsilon)) \\ &\qquad + (m_1+m_2)ga(\sin\theta_o\cos\epsilon + \cos\theta_o\sin\epsilon) = 0 \end{aligned}$$

keeping only terms linear in ϵ and simplifying yields

$$m_1a^2(\ddot{\epsilon} - \Omega^2(\sin\theta_o\cos\theta_o + (\cos^2\theta_o - \sin^2\theta_o)\epsilon)) + 2m_2a^2\ddot{\epsilon}\sin^2\theta_o + (m_1+m_2)ga(\sin\theta_o + \cos\theta_o\epsilon) = 0$$

and, by the definition of θ_o, this simplifies to

$$(m_1a^2 + 2m_2a^2\sin^2\theta_o)\ddot{\epsilon} + ((m_1+m_2)ga\cos\theta_o - m_1a^2\Omega^2\cos(2\theta_o))\epsilon = 0$$

which we recognize as the equation of motion for a simple harmonic oscillator, and we read off the frequency as

$$\omega = \sqrt{\frac{(m_1+m_2)g\cos\theta_o - m_1a\Omega^2\cos(2\theta_o)}{m_1a + 2m_2a\sin^2\theta_o}}$$

Note that if $m_1a\Omega^2\cos(2\theta_o) > (m_1+m_2)g\cos\theta_o$, then ω is imaginary and the fixed point is unstable (small perturbations lead to exponential growth).

27. Review of damped undriven and driven one dimensional harmonic oscillators

a ×2) The equation of motion for an undriven harmonic oscillator is

$$m\ddot{x} = -\lambda\dot{x} - kx$$

Use a trial solution $x(t) = e^{-ct}$, substitute in the equation, and show that there are three solutions depending on whether the oscillator is under damped, critically damped or over damped:

i) $x(t) = e^{-\Lambda t}\left[A\sin\omega t + B\cos\omega t\right]$

ii) $x(t) = e^{-\Lambda t}\left[At + B\right]$

iii) $x(t) = Ae^{\Lambda_1 t} + Be^{\Lambda_2 t}$

Find the values of the Λ's and ω for each case, in terms of m, λ and k. (Note, k is the "spring constant" as in the conservative potential $U(x) = \frac{1}{2}kx^2$, λ is the damping coefficient, and m the mass.)

Solutions:

Review Landau Chapter 5 Page 74.
Substitute e^{-ct} into the equation, we obtain

$$mc^2 - \lambda c + k = 0$$

$$c = \frac{\lambda \pm \sqrt{\lambda^2 - 4mk}}{2m}$$

There are three cases.

i) $\lambda^2 - 4mk < 0$, under damped;
ii) $\lambda^2 - 4mk = 0$, critically damped;
iii) $\lambda^2 - 4mk > 0$, over damped.

Case i), $\Lambda = \dfrac{\lambda}{2m}$, and $\omega = \dfrac{\sqrt{4mk - \lambda^2}}{2m}$, $x(t) = e^{-\Lambda t}[A'e^{i\omega t} + B'e^{-i\omega t}] = e^{-\Lambda t}\left[A\sin\omega t + B\cos\omega t\right]$.

Case ii), $\Lambda = \dfrac{\lambda}{2m}$, one solution is $e^{-\Lambda t}$, another solution will be $te^{-\Lambda t}$ which is easy to show as following:

$$\dot{x}(t) = e^{-\Lambda t} - \Lambda te^{-\Lambda t}$$

$$\ddot{x}(t) = -2\Lambda e^{-\Lambda t} + \Lambda^2 te^{-\Lambda t} = (\Lambda^2 t - 2\Lambda)e^{-\Lambda t}$$

$$m\ddot{x}(t) = m(\Lambda^2 t - 2\Lambda)e^{-\Lambda t}$$

$$-\lambda\dot{x} - kx = [(\lambda\Lambda - k)t - \lambda]e^{-\Lambda t}$$

As long as $\lambda^2 - 4mk = 0$, $m\ddot{x} = -\lambda\dot{x} - kx$.
Thus, the general solution in this case is $x(t) = e^{-\Lambda t}\left[At + B\right]$.

Case iii), $\Lambda_1 = -\dfrac{\lambda + \sqrt{\lambda^2 - 4mk}}{2m}$, $\Lambda_2 = -\dfrac{\lambda - \sqrt{\lambda^2 - 4mk}}{2m}$, $x(t) = Ae^{\Lambda_1 t} + Be^{\Lambda_2 t}$.

b) A driven damped simple harmonic oscillator obeys the equation

$$m\ddot{x} = -\lambda\dot{x} - kx + C\sin\omega t$$

and its solution has the form $x(t) = x_I(t) + x_{II}(t)$ where $x_I(t)$ is the transient solution and has the form of the solution in part a). Show that $x_{II}(t)$, the steady state solution, has the form

$$x_{II}(t) = \frac{D}{\sqrt{(\omega^2 - \omega_o^2)^2 + \Gamma^2}}\sin(\omega t + \phi)$$

and find ω_o, D, Γ and ϕ in terms of the constants describing the properties of the oscillator (m, λ and k) and the drive (C and ω).

Solutions:

We need to show that x_{II} satisfies

$$m\ddot{x}_{II} = \lambda \dot{x}_{II} - kx_{II} + C\sin\omega t$$

The strategy we take here is that we solve the equation $m\ddot{y} - \lambda\dot{y} + ky = Ce^{i\omega t}$ and take $x_{II} = \text{Im}(y)$. Substitute $y = Ae^{i(\omega t+\phi)}$ to the equation,

$$(-m\omega^2 - i\omega\lambda + k)Ae^{i(\omega t+\phi)} = Ce^{i\omega t}$$

$$e^{i\phi} = \frac{C}{A(-m\omega^2 - i\omega\lambda + k)}$$

$$\Rightarrow A = \frac{C}{\sqrt{(k-m\omega^2)^2 + \omega^2\lambda^2}} \qquad \phi = \tan^{-1}\left(\frac{\omega\lambda}{k-m\omega^2}\right)$$

$$x_{II}(t) = \text{Im}(Ae^{i(\omega t+\phi)}) = A\sin(\omega t+\phi) = \frac{C}{\sqrt{(k-m\omega^2)^2+\omega^2\lambda^2}}\sin(\omega t+\phi)$$

$$= \frac{D}{\sqrt{(\omega^2-\omega_0^2)^2+\Gamma^2}}\sin(\omega t+\phi)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$, $\phi = \tan^{-1}\left(\frac{\omega\lambda}{k-m\omega^2}\right)$, $D = \frac{C}{m}$ and $\Gamma = \frac{\omega\lambda}{m}$.

28. A driven oscillator is described by

$$\ddot{x} + \omega_o^2 x = \frac{F}{m}\cos(\gamma t + \alpha)$$

We found that the solution off resonance is

$$x(t) = B\cos(\omega_o t + \beta) + \frac{F/m}{\omega_o^2 - \gamma^2}\cos(\gamma t + \alpha).$$

which we can rearrange to

$$x(t) = C\cos(\omega_o t + \kappa) + \frac{F/m}{\omega_o^2-\gamma^2}\left(\cos(\gamma t+\alpha) - \cos(\omega_o t + \alpha)\right).$$

with new constants C and κ.

a) If the oscillator is driven close to the natural frequency ω_o, we can write $\omega_o = \gamma + \epsilon$ with $\epsilon \ll \omega_o$. Keeping terms only linear in ϵ (i.e. set any ϵ with higher power to zero), show that we can write

$$\begin{aligned} x(t) &= \cos(\omega_o t+\kappa)\left(C + \frac{F/m}{2\omega_o\epsilon}\cos(-\epsilon t + \alpha - \kappa)\right) \\ &- \sin(\omega_o t + \kappa)\left(\frac{F/m}{2\omega_o\epsilon}\sin(-\epsilon t + \alpha - \kappa)\right). \end{aligned}$$

b) Show that this evolves to the on resonance solution (LL 22.5) for $\epsilon \to 0$. Note: you may carry out the calculation using trigonometric identities or complex notation.

Note: to compare with LL 22.5, convert the above as follows:

$$C \to a, \quad F \to f, \quad \omega_0 \to \omega, \quad \kappa \to \alpha, \quad \alpha \to \beta$$

Solutions:

Use complex notation,

$$x(t) = \text{Re}\left\{Ce^{i(\omega_0 t+\kappa)} + \frac{F/m}{\omega_0^2 - (\omega_0-\epsilon)^2}e^{i((\omega_0-\epsilon)t+\alpha)}\right\} = \text{Re}\left\{Ce^{i(\omega_0 t+\kappa)} + \frac{F/m}{2\omega_0\epsilon}e^{i((\omega_0-\epsilon)t+\alpha)}\right\}$$

where we have used $\omega_0 = \gamma + \epsilon$ and ignored ϵ with higher power. Then factor out $e^{i(\omega_0 t+\kappa)}$,

$$x(t) = \text{Re}\left\{e^{i(\omega_0 t+\kappa)}(C + \frac{F/m}{2\omega_0\epsilon}e^{i(-\epsilon t+\alpha-\kappa)})\right\}$$

It is easy to derive that $\text{Re}\{ab\} = \text{Re}\{a\}\text{Re}\{b\} - \text{Im}\{a\}\text{Im}\{b\}$, then,

$$x(t) = \cos(\omega_0 t+\kappa)[C + \frac{F/m}{2\omega_0\epsilon}\cos(-\epsilon t+\alpha-\kappa)] - \sin(\omega_0 t+\kappa)[\frac{F/m}{2\omega_0\epsilon}\sin(-\epsilon t+\alpha-\kappa)]$$

Considering the complex form, the motion near resonance can be regarded as small fast oscillations of variable amplitude: $C + \frac{F/m}{2\omega_0\epsilon}e^{i(-\epsilon t+\alpha-\kappa)}$. The amplitude varies periodically with frequency ϵ (i.e., beats at frequency ϵ).

When this finally evolves to the on resonance solution, the amplitude of the oscillations increases linearly with time (see LL 22.5).

If we return to the complex form

$$x(t) = \text{Re}\left\{Ce^{i(\omega_0 t+\kappa)} + e^{i(\omega_0 t+\alpha)}\frac{F/m}{2\omega_0\epsilon}e^{-i\epsilon t}\right\}$$

and then take the limit as $\epsilon \to 0$, we find

$$x(t) = \text{Re}\left\{Ce^{i(\omega_0 t+\kappa)} - i\,e^{i(\omega_0 t+\alpha)}\frac{F}{2\,m\omega_0}t\right\} = C\cos(\omega_0 t+\kappa) + \frac{F}{2\,m\omega_0}\sin(\omega_0 t+\alpha)t$$

Note: to compare with LL 22.5, convert the above as follows:

$$C \to a, \quad F \to f, \quad \omega_0 \to \omega, \quad \kappa \to \alpha, \quad \alpha \to \beta$$

29. (×2) Determine the positions of stable equilibrium of a pendulum whose point of support, x_s, oscillates horizontally with high frequency: $x_s = a\cos(\gamma t)$, with $\gamma \gg \sqrt{g/l}$ (i.e., a horizontal Kapitza pendulum).

Solutions:

The coordinates of the pendulum's bob are

$$x = x_s + l\sin\phi$$

$$y = l(1-\cos\phi)$$

and the Lagrangian is

$$\begin{aligned} L &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy \\ &= \frac{m}{2}\left(a^2\gamma^2\sin^2\gamma t - 2a\gamma l\sin\gamma t\cos\phi\dot{\phi} + l^2\dot{\phi}^2\right) - mgl(1-\cos\phi) \\ &= \frac{m}{2}\left(l^2\dot{\phi}^2 - 2a\gamma l\sin\gamma t\cos\phi\dot{\phi}\right) + mgl\cos\phi + \text{total time derrivative} \end{aligned}$$

Thus, the equation of motion is

$$\frac{d}{dt}\left(ml^2\dot{\phi} - mal\gamma\sin\gamma t\cos\phi\right) - \left(ma\gamma l\sin\gamma t\sin\phi\dot{\phi} - mgl\sin\phi\right) = 0$$

which simplifies to

$$\ddot{\phi} + \frac{g}{l}\sin\phi - \frac{a}{l}\gamma^2\cos\gamma t\cos\phi = 0$$

To make progress, we first non-dimenstionalize the equations of motion by defining $\tau = \sqrt{\frac{g}{l}}t = \omega t$, $\lambda = a/l$ and $\nu = \gamma/\omega$, which yields

$$\frac{d^2\phi}{d\tau^2} + \sin\phi - \lambda\nu^2\cos\nu\tau\cos\phi = 0$$

and now we can apply our assumptions. We assume that $\nu \gg 1$ and $\lambda \ll 1$ such that $\nu\lambda \sim 1$. This produces a natural separation of scales ($\lambda\nu^2 \gg 1$) which we model by assuming ϕ is the superposition of two functions

$$\phi = \theta + \epsilon$$

where $\theta \sim 1$, $\epsilon \sim \lambda$, $d^2\theta/d\tau^2 \sim 1$ and $d^2\epsilon/d\tau^2 \sim \lambda\nu^2$. This means the equaitons of motion are

$$\frac{d^2\theta}{d\tau^2} + \frac{d^2\epsilon}{d\tau^2} + \sin\theta\cos\epsilon + \cos\theta\sin\epsilon - \lambda\nu^2\cos(\nu\tau)(\cos\theta\cos\epsilon - \sin\theta\sin\epsilon) = 0$$

By grouping terms of similar magnitudes, we can split this into two equations

$$\frac{d^2\epsilon}{d\tau^2} - \lambda\nu^2\cos(\nu\tau)(\cos\theta\cos\epsilon - \sin\theta\sin\epsilon) \approx \frac{d^2\epsilon}{d\tau^2} - \lambda\nu^2\cos(\nu\tau)\cos\theta = 0$$

$$\frac{d^2\theta}{d\tau^2} + \sin\theta\cos\epsilon + \cos\theta\sin\epsilon \approx \frac{d^2\theta}{d\tau^2} + \sin\theta + \cos\theta\epsilon = 0$$

In the first equation, we assome that θ is nearly constant on time scales important for the small oscillation ϵ and therefore obtain a solution for ϵ

$$\epsilon = -\lambda\cos(\nu\tau)\cos\theta$$

We then plug this into the equation for θ and obtain

$$\frac{d^2\theta}{d\tau^2} + \sin\theta + \frac{\lambda^2\nu^2}{2}\cos\theta\sin\theta = 0$$

We then look for fixed points ($d\theta/d\tau = 0$), and find them at

$$\sin\theta = 0 \quad \text{or} \quad \cos\theta = -\frac{2}{\lambda^2\nu^2}$$

To determine the stability of these fixed points, we refer to the effective potential formalism which says

$$U_{\text{eff}}(\theta) = -\cos\theta - \frac{\lambda^2\nu^2}{4}\sin^2\theta$$

and stability is determined by the condition that

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = \cos\theta + \frac{\lambda^2\nu^2}{2}\left(\cos^2\theta - \sin^2\theta\right) > 0$$

If $\theta = 0$:

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = 1 + \frac{\lambda^2\nu^2}{2} > 0$$

and this fixed point is *always* stable.

If $\theta = \pi$:

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = -1 + \frac{\lambda^2\nu^2}{2}$$

which implies this point is stable iff $\lambda\nu > \sqrt{2}$.

If $\cos\theta = -2/\lambda^2\nu^2$:

$$\frac{d^2U_{\text{eff}}}{d\theta^2} = \frac{-2}{\lambda^2\nu^2} + \frac{\lambda^2\nu^2}{2}\left(1 - 2\frac{\lambda^4\nu^4}{4}\right) = \frac{2}{\lambda^2\nu^2} - \frac{\lambda^2\nu^2}{2}$$

and these two fixed points are stable iff $\lambda\nu < \sqrt{2}$. Note that for these fixed points to be stable requires $\cos\theta < -1$, which is impossible for real θ.

30. OPTIONAL: We can write the solution to a simple harmonic oscillator as

$$\begin{aligned} x(t) &= x_o \cos\omega_o t + \frac{v_o}{\omega_o}\sin\omega_o t \\ &= x_1(t, x_o) + x_2(t, v_o). \end{aligned}$$

After a time Δt, the solution will be $x(t + \Delta t)$ which we may write

$$\begin{aligned} x_1(t + \Delta t, x_o) &= a x_1(t, x_o) + b x_2(t, v_o) \\ x_2(t + \Delta t, v_o) &= c x_1(t, x_o) + d x_2(t, v_o). \end{aligned}$$

Find a, b, c and d.

Solutions:

$$x_1(t, x_0) = x_0 \cos(\omega_0 t)$$
$$x_2(t, v_0) = \frac{v_0}{\omega_0}\sin(\omega_0 t)$$

Thus,

$$\begin{aligned} x_1(t+T, x_0) = x_0\cos\omega_0(t+T) &= x_0\cos(\omega_0 T)\cos(\omega_0 t) - x_0\sin(\omega_0 T)\sin(\omega_0 t) \\ &= \cos(\omega_0 T)x_1(t, x_0) - \frac{x_0\omega_0}{v_0}\sin(\omega_0 T)x_2(t, x_0) \\ x_2(t+T, x_0) = \frac{v_0}{\omega_0}\sin\omega_0(t+T) &= \frac{v_0}{\omega_0}\sin(\omega_0 T)\cos(\omega_0 t) + \frac{v_0}{\omega_0}\cos(\omega_0 T)\sin(\omega_0 t) \\ &= \frac{v_0}{\omega_0 x_0}\sin(\omega_0 T)x_1(t, x_0) + \cos(\omega_0 T)x_2(t, x_0) \end{aligned}$$

and we read off $a = \cos(\omega_0 T)$, $b = -(x_0\omega_0/v_0)\sin(\omega_0 T)$, $c = (v_0/\omega_0 x_0)\sin(\omega_0 T)$, and $d = \cos(\omega_0 T)$.

31. OPTIONAL: We can use a, b, c and d from the previous problem to make the matrix M such that $\vec{x}(t + \Delta t) = M\vec{x}(t)$. Find the eigenvalues of M. Take $\Delta t = 4\pi/\omega_o$ and find the eigenvectors.

Solutions:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 T) & -\frac{x_0\omega_0}{v_0}\sin(\omega_0 T) \\ \frac{v_0}{x_0\omega_0}\sin(\omega_0 T) & \cos(\omega_0 T) \end{pmatrix}$$
$$|M - \lambda I| = 0$$
$$\Rightarrow (\cos(\omega_0 T) - \lambda)^2 + \sin^2(\omega_0 T) = 0$$

If and only if

$$\omega_0 T = n\pi \quad (n \text{ is integer})$$

the eigenvalues are

$$\lambda_1 = \lambda_2 = \cos(\omega_0 T)$$

For $T = 4\pi/\omega_0$, $\lambda_1 = \lambda_2 = 1$,the eigenvectors are

$$\vec{X}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{X}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$