Massachusetts Institute of Technology

8.223, Classical Mechanics II IAP 2017

Solutions 4

32. Show that for $F(x_1, ..., x_n)$, the Legendre transformation is

$$G(s_1, ..., s_n) = \sum_{i=1}^{n} x_i s_i - F$$

where

$$s_i = \left(\frac{\partial F}{\partial x_i}\right)_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n}$$

and has the property that

$$x_i = \frac{\partial G}{\partial s_i}$$

Solutions:

We just follow the same process as in class, but with more variables. By definition $s_i = \left(\frac{\partial F}{\partial x_i}\right)_{x_1,x_2,\cdots,x_n}$ and we write out

$$dF = \sum_{i=1}^{n} s_i dx_i$$

For $G = \sum_{i=1}^{n} s_i x_i - F$, we find

$$dG = \sum_{i=1}^{n} s_i dx_i + \sum_{i=1}^{n} x_i ds_i - dF = \sum_{i=1}^{n} s_i dx_i + \sum_{i=1}^{n} x_i ds_i - \sum_{i=1}^{n} s_i dx_i = \sum_{i=1}^{n} x_i ds_i$$

Since we now have $dG = \sum_{i=1}^{n} x_i ds_i$, we have transformed to having the s_i's as coordinates, so the Legendre transform worked.

33. Starting from the Lagrangian for the simple harmonic oscillator,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2,$$

find the momentum, the Hamiltonian $H(p, q)$ and Hamilton's equations of motion.

Solutions:

From $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$, we need to get p and do the Legendre transform. First $p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$, which we solve to get $\dot{x} = \frac{p}{m}$. Now

$$H = p\dot{x} - L = p\frac{p}{m} - \frac{1}{2}m(\frac{p}{m})^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx$$

As is often the case, we get the first equation as the definition of p and the second equation as Newton's laws.

34. Continuing the previous exercise, solve Hamilton's equations and make a plot of typical trajectories in (p, q) space (referred to as "phase space".)

    Solutions:

    It's very easy to solve the Hamilton's equations in the previous problem,

    $$m\ddot{x} = \dot{p} = -kx$$

    $$\Rightarrow x = x_0 \cos(\omega t + \phi)$$

    where $\omega = \sqrt{\frac{k}{m}}$. Since the Hamiltonian doesn't explicitly depend on time, the energy is conserved, which means

    $$\frac{p^2}{2m} + \frac{1}{2}kx^2 = E \quad \text{(costant)}$$

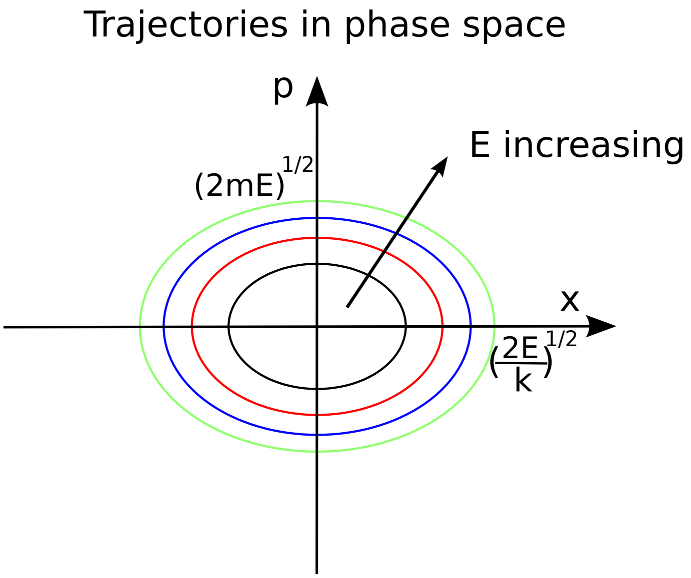

Figure 1: Trajectories in phase space

35. Give the Hamiltonian in three dimensions for a system with potential $U(r) = -\alpha/r$ using spherical polar coordinates. Write the equations of motion and identify the conserved momenta. Use this result (and a proper reference frame) to reduce the problem to one dynamical variable and associated conjugated momentum (they will be r and p_r).

Solutions:

$$L = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu(r\sin\theta\dot{\phi})^2 + \frac{1}{2}\mu(r\dot{\theta})^2 + \frac{\alpha}{r}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \mu\dot{r}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2\theta\dot{\phi}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta}$$

$$H = \dot{r}p_r + \dot{\phi}p_\phi + \dot{\theta}p_\theta - L$$

$$= \left(\frac{p_r}{\mu}\right)p_r + \left(\frac{p_\phi}{\mu r^2\sin^2\theta}\right)p_\phi + \left(\frac{p_\theta}{\mu r^2}\right)p_\theta - \frac{1}{2}\mu\left(\frac{p_r}{\mu}\right)^2 + \frac{1}{2}\mu r^2\sin^2\theta\frac{p_\phi^2}{\mu^2 r^4\sin^4\theta} + \frac{1}{2}\mu r^2\frac{p_\theta^2}{\mu^2 r^4} - \frac{\alpha}{r}$$

$$= \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2\sin^2\theta} + \frac{p_\theta^2}{2\mu r^2} - \frac{\alpha}{r}$$

Equations of motion are

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow p_\phi = M(\text{constant})$$

$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{\mu r^2\sin\theta}$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\phi^2}{\mu r^3\sin^2\theta} + \frac{p_\theta^2}{\mu r^3} - \frac{\alpha}{r^2}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{\mu}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\phi^2}{\mu r^2\sin^3\theta}\cos\theta = \frac{M^2}{\mu r^2\sin^3\theta}\cos\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}$$

As before we use the fact that the $\hat{z}$ axis may be chosen so that motion takes place in the x-y plane so that $\theta = \pi/2,\ p_\theta(0) = 0 \Rightarrow \cos\theta = 0 \Rightarrow \dot{p}_\theta = 0 \Rightarrow p_\theta = 0\ \forall\ t \Rightarrow \dot{\theta} = 0\ \forall\ t$. Then

$$\dot{p}_r = \frac{M^2}{\mu r^3} - \frac{\alpha}{r^2}$$

$$\dot{r} = \frac{p_r}{\mu}$$

36. OPTIONAL Draw phase space diagrams in r and p_r for the previous problem.

Solutions:

$$\frac{dp_r}{dt} = \frac{M^2}{\mu r^3} - \frac{\alpha}{r^2}$$

$$\frac{dr}{dt} = \frac{p_r}{\mu}$$

Thus,

$$\frac{dp_r}{dr} = \frac{\mu}{p_r}\left(\frac{M^2}{\mu r^3} - \frac{\alpha}{r^2}\right)$$

$$p_r dp_r = \left(-\frac{M^2}{r^3} + \frac{\mu\alpha}{r^2}\right) dr$$

$$\Rightarrow \frac{1}{2}p_r^2 = -\frac{M^2}{2r^2} + \frac{\mu\alpha}{r} + \text{Constant}$$

$$\Rightarrow p_r^2 = -\frac{M^2}{r^2} + \frac{2\mu\alpha}{r} + \text{C}$$

Recall the real space orbits $\frac{p}{r} = 1 + \epsilon\cos\phi$ with $p = \frac{M^2}{\mu\alpha}$ and $\epsilon = \sqrt{\frac{2M^2E}{\mu\alpha^2} + 1}$

Considering for different types of orbits we get:

(a) Circular: $\epsilon = 0$, $E = -\frac{\mu\alpha^2}{2M^2}$. $r = r_{\min} = \frac{M^2}{\mu\alpha} = p$ and $p_r = 0$. Using this, we find $C = 2\mu E$. In the phase space diagram, it's a point at $(p, 0)$.

(b) Ellipse: $0 < \epsilon < 1$, $-\frac{\mu\alpha^2}{2M^2} < E < 0$. $r_{\min} = \frac{p}{1+\epsilon}$, $r_{\max} = \frac{p}{1-\epsilon}$, at which points, $\dot{r} = 0 \Rightarrow p_r = 0$. Similarly, we find $C = 2\mu E$.

$$p_r^2 = -M^2\left(\frac{1}{r^2} - \frac{2\mu\alpha}{M^2 r}\right) + 2\mu E = -M^2\left((\frac{1}{r^2} - \frac{2}{pr}\right) + 2\mu E$$

For $-\frac{\mu\alpha^2}{2M^2} < E < 0$, $-\frac{M^2}{p^2} = -\frac{\mu^2\alpha^2}{M^2} < 2\mu E < 0$.

Let's take $E = -\frac{\mu\alpha^2}{4M^2}$ for example, $\epsilon = \frac{\sqrt{2}}{2}$,

$$p_r^2 = -M^2\left[\left(\frac{1}{r} - \frac{1}{p}\right)^2 - \frac{1}{2p^2}\right]$$

$$r_{\min} = \frac{p}{1 + \frac{\sqrt{2}}{2}}, \qquad r_{\max} = \frac{p}{1 - \frac{\sqrt{2}}{2}}$$

(c) Parabola: $\epsilon = 1$, $E = 0$. $r_{\min} = \frac{p}{2}$, at which point, $\dot{r} = 0 \Rightarrow p_r = 0$. We also get $C = 2\mu E = 0$. Thus,

$$p_r^2 = -\frac{M^2}{r^2} + \frac{2\mu\alpha}{r} = -M^2\left(\frac{1}{r^2} - \frac{2}{rp}\right)$$

with $r_{\min} = \frac{p}{2}$.

(d) Hyperbola: $\epsilon > 1$, $E > 0$. $r_{\min} = \dfrac{p}{1+\epsilon}$, at which point, $\dot{r} = 0 \Rightarrow p_r = 0$. We find $C = 2\mu E > 0$. Take $E = \dfrac{\mu\alpha^2}{2M^2}$ ($\epsilon = \sqrt{2}$) for example.

$$p_r^2 = -M^2 \left[\left(\frac{1}{r} - \frac{1}{p} \right)^2 \right] + \frac{2M^2}{p^2}$$

with $r_{\min} = \dfrac{p}{1+\sqrt{2}}$.

Finally, we get the relationship between r and p_r is

$$p_r^2 = -\frac{M^2}{r^2} + \frac{2\mu\alpha}{r} + 2\mu E$$

For different cases, we plot it in phase space by assuming $p = M = 1$ (Figure 1).

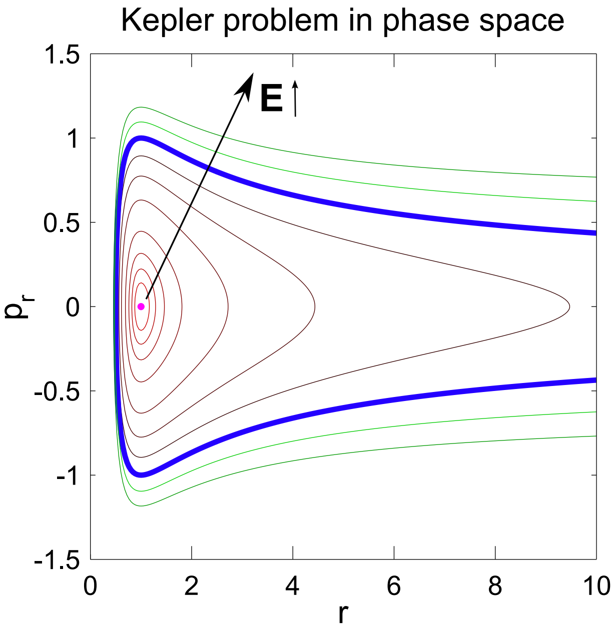

Figure 2: Kepler problem in phase space

37. Show that:

$[q_1, q_2] = 0$
$[p_1, p_2] = 0$
$[q_1, p_2] = 0$
$[p_1, q_1] = 1$
$[p_1^2, q_1] = 2p_1$
$[p_1, q_1^2] = 2q_1$

Solutions: Note to graders, this entire problem is only worth 5 points.

I'll use Landau's convention, where

$$[f, g] = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i} \frac{\partial f}{\partial q_i} \right)$$

And I'm going to make heavy use of the Kronecker delta, defined as

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$[q_1, q_2] = \sum_{i=1}^{n} \left(\frac{\partial q_1}{\partial p_i} \frac{\partial q_2}{\partial q_i} - \frac{\partial q_1}{\partial q_i} \frac{\partial q_2}{\partial p_i} \right) = \sum_{i=1}^{n} (0 \cdot \delta_{i,2} - \delta_{i,1} \cdot 0) = 0$$

$$[p_1, p_2] = \sum_{i=1}^{n} \left(\frac{\partial p_1}{\partial p_i} \frac{\partial p_2}{\partial q_i} - \frac{\partial p_1}{\partial q_i} \frac{\partial p_2}{\partial p_i} \right) = \sum_{i=1}^{n} (\delta_{i,1} \cdot 0 - 0 \cdot \delta_{i,2}) = 0$$

$$[q_1, p_2] = \sum_{i=1}^{n} \left(\frac{\partial q_1}{\partial p_i} \frac{\partial p_2}{\partial q_i} - \frac{\partial q_1}{\partial q_i} \frac{\partial p_2}{\partial p_i} \right) = \sum_{i=1}^{n} (0 - \delta_{i,1} \delta_{i,2}) = 0$$

because the δ's are never both 1.

$$[p_1, q_1] = \sum_{i=1}^{n} \left(\frac{\partial p_1}{\partial p_i} \frac{\partial q_1}{\partial q_i} - \frac{\partial p_1}{\partial q_i} \frac{\partial q_1}{\partial p_i} \right) = \sum_{i=1}^{n} (\delta_{i,1} \delta_{i,1} - 0) = 1$$

$$[p^2, q] = \left(\frac{\partial p^2}{\partial p} \frac{\partial q}{\partial q} - \frac{\partial p^2}{\partial q} \frac{\partial q}{\partial p} \right) = 2p \cdot 1 - 0 = 2p$$

$$[p, q^2] = \left(\frac{\partial p}{\partial p} \frac{\partial q^2}{\partial q} - \frac{\partial p}{\partial q} \frac{\partial q^2}{\partial p} \right) = 1 \cdot 2q - 0 = 2q$$

38. Canonical transformation:

(a) Show that the transformation on $2n$-dimensional phase space associated with a coordinate transformation on configuration space, namely:

$$q_i \to Q_i(\vec{q})$$

$$p_i \to P_i(\vec{q}, \vec{p}) = \sum_j p_j \frac{\partial q_j}{\partial Q_i}$$

is a canonical transformation.

Solutions:

As was shown in class, in order to show that a transformation is canonical it is sufficient (and probably the easiest) to demonstrate that the new coordinates satisfy the Poisson bracket as functions of the old coordinates. We first compute the easy bracket:

$$[Q_i, Q_j]_{\vec{q},\vec{p}} = \sum_{k=1}^{n} \left(\frac{\partial Q_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} - \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} \right) = 0$$

Since the Q_i are independent of $\vec{p}$. Next are the mixed PQ brackets:

$$[P_i, Q_j]_{\vec{q},\vec{p}} = \sum_{k=1}^{n} \left(\frac{\partial P_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} - \frac{\partial P_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} \right) = \sum_{k=1}^{n} \frac{\partial P_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} - 0 = \sum_{k=1}^{n} \frac{\partial q_k}{\partial Q_i} \frac{\partial Q_j}{\partial q_k} = \frac{\partial Q_j}{\partial Q_i} = \delta_{ij}$$

as desired. Finally, wee need to do the P brackets:

$$[P_i, P_j]_{\vec{q},\vec{p}} = \sum_{k=1}^{n} \left(\frac{\partial P_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} - \frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} \right) = \sum_{k=1}^{n} \left(\frac{\partial q_k}{\partial Q_i} \left(\sum_{l=1}^{n} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_j} \right) - \left(\sum_{l=1}^{n} p_l \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \right) \frac{\partial q_k}{\partial Q_j} \right)$$

Now we interchange the order of summation and note that

$$\sum_{k=1}^{n} \frac{\partial q_k}{\partial Q_i} \frac{\partial^2 q_l}{\partial q_k \partial Q_j} = \frac{\partial^2 q_l}{\partial Q_i \partial Q_j}$$

$$\sum_{k=1}^{n} \frac{\partial^2 q_l}{\partial q_k \partial Q_i} \frac{\partial q_k}{\partial Q_j} = \frac{\partial^2 q_l}{\partial Q_i \partial Q_j}$$

Thus,

$$[P_i, P_j]_{\vec{q},\vec{p}} = 0$$

as desired.

(b) On a 2-dimensional phase space, show that the transformation

$$q \to Q = \ln \left(\frac{\sin p}{q} \right)$$

$$p \to P = q \cot (p)$$

is canonical.

Solutions:

Again, we simply need to check that the new coordinates satisfy the canonical commutation relations as functions of the old coordinates. The QQ and PP brackets are trivial because the Poisson bracket is antisymmetric and so vanishes for any function of p and q. Thus, we just need to check the PQ bracket, which is

$$[P, Q] = \frac{\partial P}{\partial p} \frac{\partial Q}{\partial q} - \frac{\partial P}{\partial q} \frac{\partial Q}{\partial p} = q \left(-\csc^2(p) \right) \left(-\frac{1}{q} \right) - \cot p \frac{\cos p}{\sin p} = \frac{1 - \cos^2 p}{\sin^2 p} = 1$$

(c) Try to find the generating function $F_4(p, P)$ of the canonical transformation in (b).

Solutions:

We know from class that

$$q = -\frac{\partial F_4}{\partial p} = q(p, P) = \frac{P}{\cot p}$$

$$F_4 = -\int \mathrm{d}p \frac{P}{\cot p} + g(P) = P \ln \cos p + g(P)$$

Then

$$Q = \frac{\partial F_4}{\partial P} = \ln \cos p + g'(P) = \ln \frac{\sin p}{q} = \ln \frac{\cos p}{P}$$

So

$$g'(P) = \ln \frac{1}{P}$$

$$g(P) = P + P \ln \frac{1}{P}$$

Therefore,

$$F_4(p, P) = P \ln \cos p + P + P \ln \frac{1}{P} = P \left(1 + \ln \frac{\cos p}{P} \right).$$

39. As shown in Fig. 1, consider a mass m attached to a string which in turn is nailed to point A on a circular spool of radius R. The whole system lies on the horizontal plane, and the spool is fixed so it **cannot rotate**. As the mass slides without friction, the string remains taut and either winds or unwinds around the spool. B is the point where the string leaves the spool. Let the total length of the string be l, and let s denote the free length of the string, that is, the length from B to the mass. We align the coordinate axes so that the center O of the spool corresponds to $x = y = 0$ and the radius to OA is in the positive y direction. Let θ denote the angle between OA and OB.

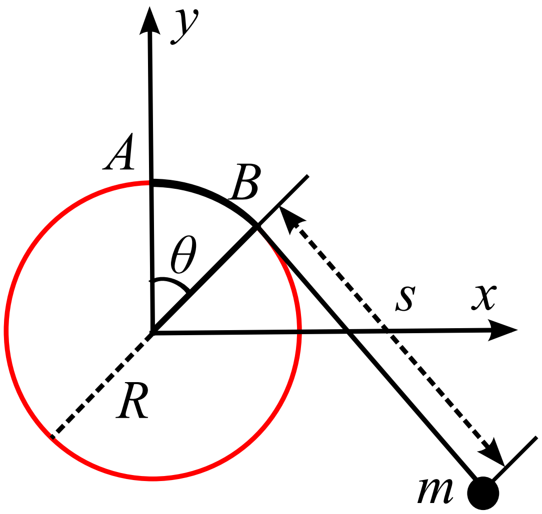

Figure 3:

(a) Express the coordinates (x, y) of the mass in terms of s, θ and R. Using the constraint relating s and θ to the total length, find the Lagrangian $L(s, \dot{s})$ of the system in terms of the dynamical coordinates s and its associated velocity $\dot{s}$.

Solutions:

The Cartesian coordinates of the position of the mass are

$$x = R\sin\theta + s\cos\theta$$

$$y = R\cos\theta - s\sin\theta$$

with the constraint

$$s + R\theta = l \rightarrow \dot{s} + R\dot{\theta} = 0 \rightarrow \dot{\theta} = -\frac{\dot{s}}{R}$$

So

$$\dot{x} = R\dot{\theta}\cos\theta - s\dot{\theta}\sin\theta + \dot{s}\cos\theta = (R\dot{\theta} + \dot{s})\cos\theta - s\dot{\theta}\sin\theta = \frac{s\dot{s}}{R}\sin\theta$$

$$\dot{y} = -R\dot{\theta}\sin\theta + s\dot{\theta}\cos\theta - \dot{s}\sin\theta = -(R\dot{\theta} + \dot{s})\sin\theta + s\dot{\theta}\cos\theta = -\frac{s\dot{s}}{R}\cos\theta$$

Since the potential energy keeps constant, the Lagrangian is just

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\frac{s^2\dot{s}^2}{R^2}$$

(b) Find the Hamiltonian $H(s,p)$, write Hamilton's equations, and confirm that $\frac{p}{s}$ is a constant of the motion. Use this information to find $s(t)$ in terms of its initial value s_0 and the total energy E.

Solutions:

$$p = \frac{\partial L}{\partial \dot{s}} = \frac{ms^2\dot{s}}{R^2} \to \dot{s} = \frac{pR^2}{ms^2}$$

$$H = p\dot{s} - L = \frac{1}{2}m\frac{s^2\dot{s}^2}{R^2} = \frac{R^2}{2m}\frac{p^2}{s^2}$$

$$\dot{s} = \frac{\partial H}{\partial p} = \frac{2H}{p}$$

$$\dot{p} = -\frac{\partial H}{\partial s} = \frac{2H}{s}$$

Then

$$\frac{d}{dt}\left(\frac{p}{s}\right) = \frac{s\dot{p} - p\dot{s}}{s^2} = \frac{s\frac{2H}{s} - p\frac{2H}{p}}{s^2} = 0$$

which means $\frac{p}{s}$ is a constant of the motion. Indeed, from the Hamiltonian

$$H = E = \frac{R^2}{2m}\frac{p^2}{s^2} \to \frac{p}{s} = \pm\frac{\sqrt{2mE}}{R}$$

We can also get see that

$$s^2\dot{s}^2 = \frac{2ER^2}{m} \to s\dot{s} = \pm\sqrt{\frac{2E}{m}}R$$

From which, we have

$$\frac{d}{dt}s^2 = \pm 2\sqrt{\frac{2E}{m}}R \to s^2(t) = \pm 2\sqrt{\frac{2E}{m}}Rt + s_0^2$$

(c) Consider the following change of coordinates:

$$Q = s^2, \;\; P = \frac{p}{\lambda s}$$

Find the value of the constant λ so that the transformation is canonical and give $H'(Q,P)$.

Solutions:

$$[P,Q] = \frac{\partial P}{\partial p}\frac{\partial Q}{\partial s} - \frac{\partial P}{\partial s}\frac{\partial Q}{\partial p} = \frac{1}{\lambda s}(2s) = 1$$

Thus, λ has to be 2. And $H' = \frac{R^2}{2m}(2P)^2 = \frac{2}{m}R^2P^2$.

40. For a Lagrangian with a velocity dependent potential

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \lambda y\dot{x},$$

a) Show that the solutions have the form

$$x(t) = -R\cos\left(\frac{\lambda}{m}t + \phi\right) + x_o$$

$$y(t) = R\sin\left(\frac{\lambda}{m}t + \phi\right) + y_o.$$

Solutions:

Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = m\ddot{x} - \lambda\dot{y} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = m\ddot{y} + \lambda\dot{x} = 0$$

Differentiating one of these equations and plugging it into the other yields:

$$\dddot{x} = \frac{\lambda}{m}\ddot{y} = -\frac{\lambda^2}{m^2}\dot{x} \Rightarrow \dddot{x} + \frac{\lambda^2}{m^2}\dot{x} = 0$$

Then,

$$x(t) = -R\cos\left(\frac{\lambda}{m}t + \phi\right) + x_0$$
$$\ddot{y}(t) = -\frac{\lambda}{m}\dot{x}(t) = -\left(\frac{\lambda}{m}\right)^2 R\sin\left(\frac{\lambda}{m}t + \phi\right)$$
$$\Rightarrow y = R\sin\left(\frac{\lambda}{m}t + \phi\right) + y_0$$

b) Find the conjugate momenta.

Solutions:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \lambda y = mR\frac{\lambda}{m}\sin\left(\frac{\lambda}{m}t + \phi\right) - \lambda\left(R\sin(\frac{\lambda}{m}t + \phi) + y_0\right) = -\lambda y_0$$
$$p_y = \frac{\partial L}{\partial \dot{y}} = m\frac{\lambda}{m}R\cos\left(\frac{\lambda}{m}t + \phi\right) = \lambda R\cos\left(\frac{\lambda}{m}t + \phi\right)$$

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January IAP 2017